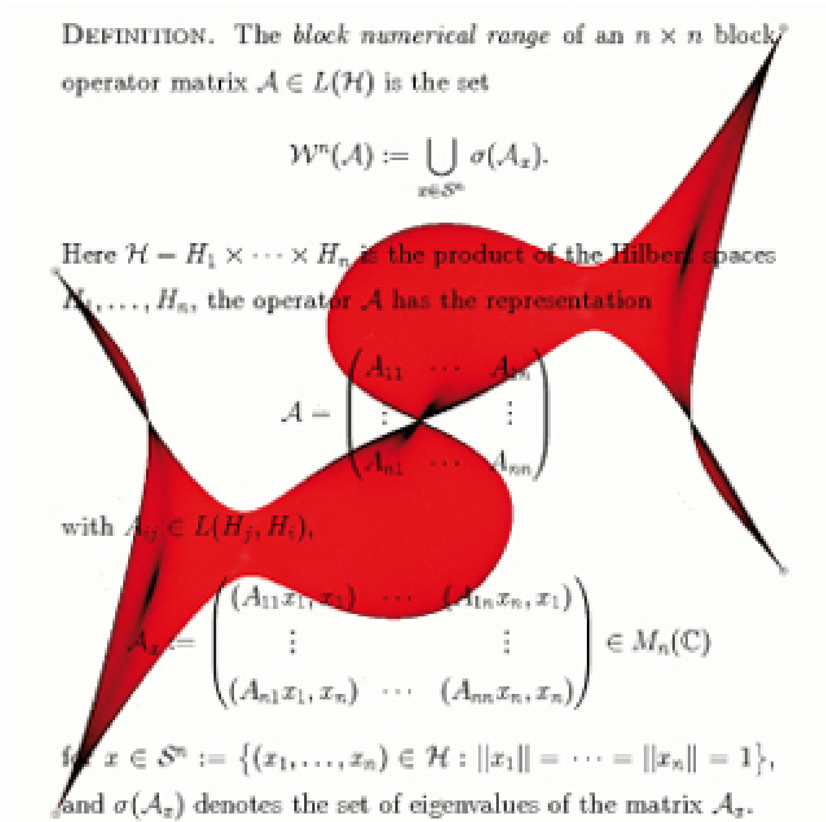
Christiane Tretter

# Spectral Theory of Block Operator Matrices and Applications



DEFINITION. The *block numerical range* of an $n \times n$ block operator matrix $\mathcal{A} \in L(\mathcal{H})$ is the set

$$\mathcal{W}^n(\mathcal{A}) := \bigcup_{x \in \mathcal{S}^n} \sigma(\mathcal{A}_x).$$

Here $\mathcal{H} = H_1 \times \cdots \times H_n$ is the product of the Hilbert spaces $H_1, \dots, H_n$, the operator $\mathcal{A}$ has the representation

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

with $A_{ij} \in L(H_j, H_i)$,

$$\mathcal{A}_x := \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) \end{pmatrix} \in M_n(\mathbb{C})$$

for $x \in \mathcal{S}^n := \{(x_1, \dots, x_n) \in \mathcal{H} : \|x_1\| = \cdots = \|x_n\| = 1\}$, and $\sigma(\mathcal{A}_x)$ denotes the set of eigenvalues of the matrix $\mathcal{A}_x$.

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**SPECTRAL THEORY OF BLOCK OPERATOR MATRICES AND APPLICATIONS**



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# Preface

Block operator matrices are matrices the entries of which are linear operators between Banach or Hilbert spaces. They arise in various areas of mathematics and its applications: in systems theory as Hamiltonians (see [CZ95]), in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns (see [SAD+00]), in saddle point problems in non-linear analysis (see [BGL05]), in evolution problems as linearizations of second order Cauchy problems (see [EN00]), and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in mathematical physics, *e.g.* in fluid mechanics (see [Cha61]), magnetohydrodynamics (see [Lif89]), and quantum mechanics (see [Tha92]). In all these applications, the spectral properties of the corresponding block operator matrices are of vital importance as they govern for instance the time evolution and hence the stability of the underlying physical systems.

The aim of this book is to present a wide panorama of methods to investigate the spectral properties of block operator matrices. Particular emphasis is placed on classes of block operator matrices to which standard operator theoretical methods do not readily apply: non-self-adjoint block operator matrices, block operator matrices with unbounded entries, non-semi-bounded block operator matrices, and classes of block operator matrices arising in mathematical physics. The main topics include:

- localization of the spectrum and investigation of its structure,
- description of the essential spectrum,
- characterization and estimates of eigenvalues,
- block diagonalization and invariant subspaces,
- solutions of algebraic Riccati equations,
- applications to concrete problems from mathematical physics.

We shall address these problems using the particular block structure of the operators and the properties of their operator entries. The methods employed come from a number of different areas:

- the theory of linear operators (numerical ranges, perturbation theory),
- classical functional analysis (fixed point theorems),
- complex analysis (analytic operator functions, factorization theorems).

The book gives an account on recent research on the spectral theory of block operator matrices and its applications in mathematical physics. It contains results which were published roughly during the last 15 years. A number of theorems, however, was found while this book was written and are still in the process of being published. The three chapters may be of interest for different groups of readers: Chapter 1 deals exclusively with bounded block operator matrices and contains methods and results which are of interest even in the matrix case. Chapter 2, which may be read independently of Chapter 1, is focused on unbounded block operator matrices and is particularly suited for applications to differential operators. Chapter 3 contains applications of the results of Chapter 2 to various spectral problems from mathematical physics.

No particular focus is placed on block operator matrices arising in systems theory and in evolution equations. Although some of the methods presented may be applicable, it seems to be impossible for a single author and a single book to cover also the vast range of results in these two important areas; nevertheless, certain points of intersection will be mentioned.

This book could not have be written without the contributions and help of many people. Sincere thanks go to my coauthors and friends Vadim Adamjan, Margarita Kraus, Heinz Langer, Matthias Langer, Alexander Markus, Marco Marletta, and Volodya Matsaev; my teacher Reinhard Mennicken; my former PhD students Markus Wagenhofer (with special thanks for providing the beautiful figures and the cover), Monika Winklmeier, and Christian Wyss; my present PhD students Jean-Claude Cuenin and Jan Nesemann; my colleague Alexander Motovilov; Deutsche Forschungsgemeinschaft (DFG, Germany) and Engineering and Physical Sciences Research Council (EPSRC, UK) for their most valuable funding; and, finally, to Zhang Ji and Jessie Tan from World Scientific for all their support and patience.

Bern, July 2008 Christiane Tretter

# Introduction

As an introduction, we first describe the historical background of the spectral theory of linear operators. In the second part, the state of the art of research on the spectral theory of block operator matrices is outlined. The third and last part contains a brief description of the contents of this book.

**1. Historical background.** The spectral theory of linear, in particular self-adjoint, operators in a Hilbert space is one of the major advances in mathematics of the 20th century (see [Ste73a] for a historical survey). It was initiated by D. Hilbert in his famous six papers on integral equations from 1904 to 1910 (see [Hil53]) which contain all basic ideas for the spectral theorem for bounded self-adjoint operators. Simultaneously, H. Weyl further advanced the theory for singular second order ordinary differential equations in his seminal paper [Wey10]. The next major breakthrough came during the years 1927 to 1929 when J. von Neumann developed the abstract concept of Hilbert space and linear operators therein and initiated the spectral theory for unbounded self-adjoint operators (see [vN30a]). Von Neumann's work was driven by the needs of quantum mechanics (see [vN27], [vN32]), which was created in 1925/26 mainly by W. Heisenberg, E. Schrödinger, and P. Dirac (see [BJ25], [BHJ26], [Sch26a], [Sch26b], [Dir25], [Dir26]). Between 1927 and 1932, this spectral theory was elaborated and extended to unbounded normal operators by F. Riesz (see [Rie30]) and, in great detail and independently of further work by von Neumann (see [vN30b]), by M.H. Stone (see [Sto32]). Further important contributions concerning extensions of semi-bounded symmetric operators and applications to differential operators are due to K.O. Friedrichs in 1934 (see [Fri34]) and M.G. Kreĭn in 1947 (see [Kre47b], [Kre47a]).

Another important field that was stimulated by problems from mathematical physics is the perturbation theory of linear operators (see [Kat95]).

It originates in the works of Lord Rayleigh from 1877 on vibrating systems (see [Ray26]) and of E. Schrödinger from 1926 on eigenvalue problems in quantum mechanics. The first important contribution in this field is due to H. Weyl in 1909 and concerns the stability of the now so-called essential spectrum of a bounded self-adjoint operator (see [Wey09]). A mathematically rigorous perturbation theory for eigenvalues and eigenvectors of self-adjoint operators $T(\kappa)$ depending analytically on a parameter $\kappa$ was developed by F. Rellich between 1937 and 1942 in a series of fundamental papers (see [Rel42]). Later, in 1948, K.O. Friedrichs created a perturbation theory for the continuous spectrum which proved to be useful in scattering theory and quantum field theory (see [Fri48], [Fri52]). In 1949 T. Kato started his deep investigations on the perturbation theory of linear operators (see [Kat50], [Kat52], [Kat53]) which form one of the bases of his famous so entitled 1966 monograph [Kat95].

The spectral theory of non-self-adjoint linear operators is much more involved than that of self-adjoint operators. Although the first pioneering works of G.D. Birkhoff on eigenfunction expansions for non-self-adjoint differential operators (see [Bir08], [Bir12], [Bir13]) were written almost at the same time as Hilbert's famous six papers, it took about 40 years for abstract results to follow. Since then, a wealth of results has become available in the literature which are not so widely known, in particular, among physicists and engineers. It is impossible to give even a brief account of them and so only a few milestones can be mentioned. In the years 1951/52 B. Sz.-Nagy, F. Wolf and T. Kato generalized Rellich's results and obtained the first theorems on the perturbation theory of closed linear operators (see [SN51], [Wol52], [Kat52]). At the same time, in 1951, M.V. Keldysh's cornerstone work on the completeness of eigenvectors and associated vectors and eigenvalue asymptotics of non-self-adjoint operator polynomials was published (see [Kel51]), which had a great impact on the spectral theory of non-self-adjoint differential operators. A seminal work from 1957 on the stability of various spectral properties, in particular, of the index for non-self-adjoint operators is due to I.C. Gohberg and M.G. Kreĭn (see [GK60]). Almost simultaneously and independently, T. Kato established similar results (see [Kat58]). I.C. Gohberg and M.G. Kreĭn also wrote a comprehensive account of results on non-self-adjoint operators as early as 1965 (see [GK69]). One focus of this book is on completeness results as initiated by V.I. Matsaev (see *e.g.* [Mat61], [Mat64]) and by V.B. Lidskiĭ (see *e.g.* [Lid59b], [Lid59a]); it also contains many other important contributions, *e.g.* on classes of compact linear operators and $s$-numbers, estimates of the growth of the resolvent of non-self-adjoint linear operators, and perturbation determinants. Another important direction was opened up in 1954 by N. Dunford who developed the theory of spectral operators (see [Dun54]); a detailed presentation of the latter and an enormous collection of results on non-self-adjoint operators is contained in the volume [DS88] by N. Dunford and J.T. Schwartz. A structure theory for non-self-adjoint operators was created in 1954 by M.S. Livšic; in [Liv54] he introduced the notions of operator colligations (or nodes, following the literal translation from Russian) and characteristic functions and employed them intensively with his colleagues, in particular, M.S. Brodskiĭ (see [Bro71], [BL58], and the monograph [LY79] with its review in [Bal81]). In the 1960ies, the notion of characteristic functions appeared also in the work of B. Sz.-Nagy and C. Foiaş in the different context of unitary dilation spaces for contractions (see [SNF70]). One of the main tools in almost all of the above works is the extensive use of the theory of functions, either for studying the behaviour of the resolvents of non-self-adjoint operators or of operator colligations by means of the factorization of characteristic functions (see *e.g.* the monograph [BGK79] by H. Bart, I.C. Gohberg, and M.A. Kaashoek).

Linear operators that are self-adjoint with respect to an indefinite inner product were brought up in quantum field theory, in works from 1942/43 by P. Dirac (see [Dir42]) and W. Pauli (see [Pau43]). The first basic result for operators in infinite dimensional spaces with indefinite inner product is due to L.S. Pontryagin in 1944 (see [Pon44]). Inspired by a mechanical problem considered by S.L. Sobolev (which was, at that time, secret military research and published only in 1960, see [Sob60]), Pontryagin proved the existence of a special invariant subspace for a self-adjoint operator in a space with finite-dimensional positive part (now called Pontryagin space). This result was extended and generalized by I.S. Iohvidov, M.G. Kreĭn, and H. Langer between 1956 and 1962 (see [IK56], [IK59], [Lan62], and the joint monograph [IKL82]). In 1963 M.G. Kreĭn and H. Langer established a spectral function for self-adjoint operators in Pontryagin spaces (see [KL63]). Shortly after, in 1965, a comprehensive spectral theory for definitizable self-adjoint operators in Krein spaces (where positive and negative part may be infinite dimensional) was set up by H. Langer (see [Lan65], [Lan82]). Many of the above results may be found in the monographs [Bog74] by J. Bognár and [AI89] by T.Ya. Azizov and I.S. Iohvidov.

In spite of all its inherent problems, the spectral theory of non-self-adjoint linear operators is of utmost importance for applications: Self-adjointness is an intrinsic property of energy-preserving (so-called closed) systems; however, in practice, many systems *e.g.* from engineering are not energy-preserving and have a non-self-adjoint state operator (so-called open systems). The recent book [TE05] by L.N. Trefethen and M. Embree on the behaviour of non-normal matrices and operators provides striking evidence of this by mentioning 19 fields with more than 8000 representative citations and by its own bibliography comprising 851 references. The significance of the problems arising due to the lack of self-adjointness, especially with regard to numerical calculations, have now become widely accepted, at least among mathematicians. Major contributions to this are due to F. Chatelin who further developed Kato's perturbation theory in view of applications to the numerical spectral analysis in 1983 (see the monograph [Cha83]). Other important concepts are pseudospectra and spectral value sets originating in works of H.J. Landau from 1975 (see [Lan75]), S.K. Godunov *et al.* from 1990 (see [GKK90]), N. Trefethen from 1992 (see [Tre92] and the monograph [TE05]), and of D. Hinrichsen and A. Pritchard in view of uncertain systems (see [HP92] and the monograph [HP05]). Important contributions to pseudospectra of differential operators are due to E.B. Davies (see [Dav99], [Dav00], and the recent survey [Dav07]).

**2. Spectral theory of block operator matrices.** If $\mathcal{A}$ is a bounded linear operator in a Hilbert space $\mathcal{H}$ and a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ is given, then $\mathcal{A}$ always admits a block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{I}$$

with linear operators $A$, $B$, $C$, and $D$ acting in or between the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For an unbounded linear operator $\mathcal{A}$ in $\mathcal{H}$, its domain $\mathcal{D}(\mathcal{A})$ need not be decomposable as $\mathcal{D}_1 \oplus \mathcal{D}_2$ with $\mathcal{D}_1 \subset \mathcal{H}_1$, $\mathcal{D}_2 \subset \mathcal{H}_2$ and hence it is an additional assumption that $\mathcal{A}$ has a representation (I) such that $\mathcal{D}(\mathcal{A}) = \big(\mathcal{D}(A) \cap \mathcal{D}(C)\big) \oplus \big(\mathcal{D}(B) \cap \mathcal{D}(D)\big)$. In this case, we call $\mathcal{A}$ an unbounded block operator matrix.

Our aim is to use information about the entries $A$, $B$, $C$, and $D$ to investigate various spectral properties of the block operator matrix $\mathcal{A}$. If $\mathcal{A}$ is not self-adjoint or symmetric and/or all entries of $\mathcal{A}$ are unbounded, we face a number of difficulties:

a) The results on self-adjoint operators and perturbations of them rely heavily on the following properties: if $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$ is self-adjoint in a Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)$, then its numerical range $W(\mathcal{A}) = \big\{(\mathcal{A}x, x) : x \in \mathcal{D}(\mathcal{A}), \|x\| = 1\big\}$ and its spectrum $\sigma(\mathcal{A})$ are real, the norm of the resolvent $(\mathcal{A}-\lambda)^{-1}$ is bounded by $1/\mathrm{dist}\big(\lambda, \sigma(\mathcal{A})\big)$ for $\lambda \notin \sigma(\mathcal{A})$, all eigenvalues are algebraically simple (*i.e.* there are no Jordan chains), and, if the spectrum of $\mathcal{A}$ is discrete (*i.e.* if it consists only of isolated eigenvalues with finite multiplicities), then $\mathcal{H}$ has an orthonormal basis of eigenvectors. For a non-self-adjoint linear operator $\mathcal{A}$ all these properties may fail: the numerical range and the spectrum need no longer be real, there are no estimates of the resolvent in terms of the spectrum (which may result in an extreme sensitivity of the spectrum to perturbations), the eigenvalues need not be algebraically simple, and, if the spectrum is discrete, then the system of eigenvectors and associated vectors may not be complete and may fail to be "linearly independent".

b) For the spectral theory of unbounded linear operators, it has to be required that the operator is closed or at least closable. However, the sum of closed or closable operators need not be closed or closable, respectively; similarly, the sum of self-adjoint operators need not be self-adjoint. As a consequence, an unbounded block operator matrix need not be closed even if so are its entries; if it is closable, then the closure need not have a block operator matrix representation. We identify classes of closable block operator matrices and describe their closures. This classification is based on inclusion relations between the domains of the entries: diagonally dominant, off-diagonally dominant, and upper dominant block operator matrices. It turns out that, in many respects, these classes require different approaches.

c) The most powerful tool to investigate all kinds of spectral data of a self-adjoint operator is the spectral function. Nothing comparable exists in the non-self-adjoint case. For bounded isolated parts of the spectrum, a contour integral over the resolvent, the so-called Riesz projection, can be used to identify a corresponding spectral subspace. However, if the spectrum consists of two unbounded isolated parts it is not clear if the spectrum splits at infinity, *i.e.* if there exist corresponding spectral subspaces which reduce the operator. Moreover, even for self-adjoint operators that are non-semi-bounded and hence have spectrum tending to $\infty$ and $-\infty$, the classical variational principles do not apply to eigenvalues in gaps of the spectrum.

d) For $2 \times 2$ matrices, the eigenvalues can be characterized as the zeroes of a quadratic polynomial, the characteristic determinant. Since, in general, the entries $A$, $B$, $C$, and $D$ of a block operator matrix (I) act between different spaces, it is not possible to multiply them in a way resembling the scalar $2 \times 2$ case; *e.g.* if $\mathcal{H}_1 = \mathbb{C}^k$ and $\mathcal{H}_2 = \mathbb{C}^l$ with $k \neq l$, then the product of matrices $AD - BC$ cannot be formed. To study the spectrum of $2 \times 2$ block operator matrices, one therefore has to find other functions encoding the spectral data (*e.g.* Schur complements). These functions depend on the complex spectral parameter and their values are linear operators; how to build them, may depend on the invertibility of the entries of the block operator matrix and on inclusions between their domains.

The above mentioned problems have been attacked in the theory of block operator matrices by various different methods:

2.1. *Quadratic numerical range and block numerical ranges.* The numerical range $W(\mathcal{A}) = \{(\mathcal{A}x, x) : x \in \mathcal{D}(\mathcal{A}),\ \|x\| = 1\}$ of a linear operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is a reliable method to localize its spectrum. It was first studied by O. Toeplitz in 1918 (see [Toe18]); he proved that the numerical range of a matrix contains all its eigenvalues and that its boundary is a convex curve. In 1919 F. Hausdorff showed that indeed the set $W(\mathcal{A})$ is convex (see [Hau19]). In fact, it turned out that this continues to hold for general bounded linear operators and that the spectrum is contained in the closure $\overline{W(\mathcal{A})}$ (see [Win29]). For unbounded linear operators $\mathcal{A}$, the inclusion of the spectrum prevails if every component of $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ contains at least one point of the resolvent set of $\mathcal{A}$; moreover, the resolvent estimate $\|(\mathcal{A} - \lambda)^{-1}\| \leq 1/\text{dist}\big(\lambda, W(\mathcal{A})\big)$ holds for $\lambda \notin \overline{W(\mathcal{A})}$ (see [Kat95]). Another interesting property is that every corner of $\overline{W(\mathcal{A})}$ belongs to the spectrum and is an eigenvalue if it lies in $W(\mathcal{A})$ (see [Don57], [Hil66]).

At first sight, the convexity of the numerical range seems to be a useful property, *e.g.* to show that the spectrum of an operator lies in a half plane. However, the numerical range often gives a poor localization of the spectrum and it cannot capture finer structures such as the separation of the spectrum in two parts. In view of these shortcomings, the new concept of quadratic numerical range was introduced in 1998 in [LT98] and further studied in [LMMT01], [LMT01]. If $\mathcal{A}$ is a bounded linear operator, a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the Hilbert space is given, and (I) is the corresponding block operator matrix representation of $\mathcal{A}$, then the quadratic numerical range $W^2(\mathcal{A})$ of $\mathcal{A}$ is the set of all eigenvalues of the $2 \times 2$ matrices

$$\mathcal{A}_{x,y} = \begin{pmatrix} (Ax, x) & (By, x) \\ (Cx, y) & (Dy, y) \end{pmatrix}, \quad x \in \mathcal{H}_1,\ y \in \mathcal{H}_2,\ \|x\| = \|y\| = 1. \tag{II}$$

The obvious generalization to $n \times n$ block operator matrices is called block numerical range of $\mathcal{A}$ (see [Wag00], [TW03]); for unbounded block operator matrices $\mathcal{A}$, only matrices $\mathcal{A}_{x,y}$ with normed elements $x \in \mathcal{D}(A) \cap \mathcal{D}(C)$ and $y \in \mathcal{D}(B) \cap \mathcal{D}(D)$ are considered (see [LT98], [Tre08]).

The quadratic numerical range is always contained in the numerical range: $W^2(\mathcal{A}) \subset W(\mathcal{A})$. However, unlike the numerical range, the quadratic numerical range is no longer convex; it consists of at most two components which need not be convex either. On the other hand, the quadratic numerical range shares many other properties with the numerical range: it enjoys the spectral inclusion property; it furnishes a resolvent estimate of the form $\|(\mathcal{A}-\lambda)^{-1}\| \leq 1/\mathrm{dist}\big(\lambda, W^2(\mathcal{A})\big)^2$ for $\lambda \notin \overline{W^2(\mathcal{A})}$; a corner of $\overline{W^2(\mathcal{A})}$ belongs to the spectrum of $\mathcal{A}$ or one of its diagonal entries $A$, $D$. Analogous results hold for the block numerical range and some, *e.g.* the spectral inclusion, also for certain classes of unbounded block operator matrices. Compared to other spectral enclosures of Gershgorin or Brauer type (see [Ger31], [Bra58]), the quadratic numerical range has the advantage of not requiring any norms of inverses.

Besides the spectral inclusion, the most important feature of the quadratic numerical range is that it yields a criterion for block diagonalizability. The corresponding theorem generalizes the well-known fact that every $2\times 2$ matrix with two distinct eigenvalues can be diagonalized: If the closure of $W^2(\mathcal{A})$ consists of two disjoint components, then $\mathcal{A}$ can be block diagonalized. An analogue for the block numerical range was proved recently under some additional conditions (see the PhD thesis [Wag07]).

Although the quadratic numerical range is a relatively recent concept, applications of it have already appeared in the literature. V.V. Kostrykin, K.A. Makarov, A.K. Motovilov used it to prove perturbation results for spectra and spectral subspaces of bounded self-adjoint operators (see [KMM07]); some of their results are presented in Section 1.3 (see Theorems 1.3.6, 1.3.7). In [Lin03] H. Linden applied the quadratic numerical range to derive enclosures for the zeroes of monic polynomials. K.-H. Förster and N. Hartanto developed a Perron-Frobenius theory for the block numerical range of (entrywise) nonnegative matrices in [FH08], thus generalizing corresponding results for the spectrum and the numerical range.

2.2. *Schur complements and factorization.* Schur complements were first used in the theory of matrices. The idea to associate the matrix $D-CA^{-1}B$ with a block matrix $\mathcal{A}$ as in (I) (with non-singular $A$) may be traced back at least to Schur (see [Sch17], and maybe even to earlier work by J. Sylvester). The name "Schur complement" was created by E. Haynsworth in 1968 when she began to study partitioned matrices (see [Hay68]). In Hilbert spaces, Schur complements may be found first in M.G. Kreĭn's famous paper [Kre47b] on the extension of self-adjoint operators, under the name "shorted operators". Apart from their numerous applications in matrix theory and numerical linear algebra, Schur complements are used in many other areas including statistics, electrical engineering, $C^*$-algebras (see *e.g.* [Zha05] with its exhaustive bibliography, [Bha07], and [CIDR05]), and in mathematical systems theory, where they appear as transfer functions of state space realizations of linear time invariant systems (see [BGKR05]).

In the theory of bounded and unbounded block operator matrices, Schur complements are powerful tools to study the spectrum and various spectral properties. This was first recognized by R. Nagel in a series of papers starting in 1985 (see [Nag85], [Nag89], [Nag90], [Nag97]). He began to develop a "matrix theory" for unbounded operator matrices with "diagonal domain" (block operator matrices in our terminology) and with "non-diagonal domain" (allowing for some coupling between the two components of elements of the domain). The intimate relation between the spectral properties of the block operator matrix $\mathcal{A}$ and those of its Schur complements

$$S_1(\lambda) = A - \lambda - B(D-\lambda)^{-1}C, \quad \lambda \notin \sigma(D),$$
$$S_2(\lambda) = D - \lambda - C(A-\lambda)^{-1}B, \quad \lambda \notin \sigma(A),$$

is obvious from the so-called (formal) Frobenius-Schur factorizations, *e.g.*

$$\mathcal{A} - \lambda = \begin{pmatrix} I & 0 \\ C(A-\lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} A-\lambda & 0 \\ 0 & S_2(\lambda) \end{pmatrix} \begin{pmatrix} I & (A-\lambda)^{-1}B \\ 0 & I \end{pmatrix}, \quad \lambda \notin \sigma(A),$$

and the corresponding factorization for the resolvent $(\mathcal{A}-\lambda)^{-1}$. Important milestones in this direction are: the paper [ALMS94] by F.V. Atkinson, H. Langer, R. Mennicken, and A.A. Shkalikov from 1994 where the essential spectrum of a upper dominant block operator matrix was determined by means of Schur complements; the paper [AL95] by V.M. Adamjan and H. Langer from 1995 which contains the key ideas for the block diagonalization of operator matrices with self-adjoint separated diagonal entries $D \ll 0 \ll A$ and bounded corners $B$, $C$ that are self-adjoint ($C = B^*$) or $\mathcal{J}$-self-adjoint ($C = -B^*$). In the subsequent papers [ALMS96], [Shk95], [AMS98], the approach of [AL95] was extended to self-adjoint block operator matrices with unbounded entries, and in [LT98] to non-self-adjoint diagonal entries $A$, $D$ with spectra separated by a vertical strip and $C = B^*$. The Schur complement approach of [ALMS94] to determine the essential spectrum was further developed by A.A. Shkalikov (see [Shk95]) and A. Jeribi *et al.* (see [MDJ06], [DJ07]) in the Banach space case, and, in the Hilbert space case, by A.Yu. Konstantinov *et al.* (see [Kon96], [Kon97], [Kon98], [Kon02], [KM02], [AK05]) who also studied the absolutely continuous spectrum and gave applications to upper dominant singular matrix differential operators.

Like Livšic' characteristic functions, Schur complements are operator-valued analytic functions reflecting the spectral properties of the associated block operator matrix $\mathcal{A}$. Also here, methods of complex analysis such as the factorization theorems by A.I. Virozub and V.I. Matsaev for the self-adjoint case (see [VM74]) and by A.S. Markus and V.I. Matsaev for the general case (see [MM75]) may be employed. The former was used by R. Mennicken and A.A. Shkalikov (see [MS96]) who generalized the results of [AL95], [ALMS96] for self-adjoint block operator matrices with $D \leq A$ to the case where $A$ and the Schur complement $S_2$ satisfy a certain separation condition. The factorization theorem by Markus and Matsaev, together with Brouwer's fixed point theorem, was used in [LMMT01] to prove the theorem on block diagonalization for bounded non-self-adjoint block operator matrices. It was shown that if the closure of the quadratic numerical range consists of two disjoint components, $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$, then the Schur complements admit factorizations

$$S_j(\lambda) = M_j(\lambda)(Z_j - \lambda), \quad j = 1, 2, \tag{III}$$

with operator functions $M_j$ that are holomorphic in $\mathcal{F}_j$ and have boundedly invertible values and linear operators $Z_j$ such that $\sigma(Z_j) = \sigma(\mathcal{A}) \cap \mathcal{F}_j$. As a consequence of the factorization (III), the block operator matrix $\mathcal{A}$ turns out to be similar to the block diagonal matrix $\operatorname{diag}(Z_1, Z_2)$.

Interestingly, in 1990/1991, before the abstract methods described above were developed, eigenvalue problems for second order differential expressions depending on the spectral parameter rationally (with so-called "floating singularity", see [Bog85]) were studied (see [LMM90], [FM91], [ALM93]). In fact, the corresponding linear operators are the Schur complements of upper dominant matrix differential operators. These investigations, especially on $\lambda$-rational Sturm-Liouville problems on compact intervals were further elaborated to study also eigenvalue accumulation and embedded eigenvalues by means of Sturm's comparison and oscillation theories (see [ALM93], [MSS98], [Lan00], [ALL01], [Lan01]). The case of singular intervals was treated by J. Lutgen in [Lut99]; the case of the whole axis with periodic boundary conditions was considered by R.O. Hryniv, A.A. Shkalikov, and A.A. Vladimirov in [HSV00], [HSV02]. It is impossible to give an account of the vast literature on matrix differential operators studied purely by means of techniques from the theory of differential equations; we only mention Dirac systems or Schrödinger type operator matrices.

In mathematical physics, Schur complements were first used by H. Feshbach in 1958 (see [Fes58]) and have since become valuable tools under the name Feshbach maps or decimation maps (see *e.g.* [Bac01, Section 7], [Lut04], [BCFS03], [GH08]). Here the entry $A$ corresponds *e.g.* to low energy states of the system without interaction and the operator $A - BD^{-1}C$ is called decimated Hamiltonian. The fact that the spectrum and eigenvalues of a block operator matrix outside the spectrum of the diagonal entry $A$, say, coincide with the spectrum and eigenvalues, respectively, of its first Schur complement $S_1$ and the relation $P_1(\mathcal{A}-\lambda)^{-1}P_1 = S_1(\lambda)^{-1}$ are referred to as the Feshbach projection method or Grushin problem in the physics literature (see [BFS98, Chapter II] and [SZ03]).

2.3. *Algebraic Riccati equations.* There are two algebraic Riccati equations formally associated with a block operator matrix $\mathcal{A}$ as in (I):

$$K_1BK_1 + K_1A - DK_1 - C = 0, \quad K_2CK_2 + K_2D - AK_2 - B = 0. \quad \text{(IV)}$$

Even in the matrix case, the existence of solutions to such quadratic operator equations is a non-trivial problem. In the following we describe a purely analytical approach which relies on rewriting the Riccati equations as operator Sylvester equations (sometimes also called Kreĭn-Rosenblum equations) and using a fixed point argument; subsequently, we present methods that are based on the close relation of solutions of Riccati equations and invariant graph subspaces of the block operator matrix.

The starting point for the fixed point approach is to write *e.g.* the first Riccati equation in the equivalent form $KA-DK = Y$ with $Y := C-KBK$. Solutions $K$ to such operator equations in integral form seem to have been found first by M.G. Kreĭn in 1948 (see [Phó91]), and later independently by Yu. Daleckiĭ (see [Dal53]) and M. Rosenblum (see [Ros56]). The key condition is that the spectra of the operator coefficients $A$ and $D$ on the left hand side have to be disjoint; then the solution is given by the so-called Daleckiĭ-Kreĭn formula (see [DK74a, Theorem I.3.2] or [GGK90, Theorem I.4.1])

$$K = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_D} (D-z)^{-1} Y (A-z)^{-1} =: \Phi(K);$$

here $\Gamma_D$ is a Cauchy contour around $\sigma(D)$ separating it from $\sigma(A)$. Since $Y = C - KBK$, this formula may be viewed as a fixed point equation for $K$. To ensure that the mapping $\Phi$ so defined is a contraction, smallness conditions have to be imposed on the coefficients $B$ and $C$. As a result of Banach's fixed point theorem, we obtain the existence and uniqueness of contractive solutions of the Riccati equation and a fixed point iteration converging to the solution in the operator norm.

The first to use a fixed point argument in connection with a Kreĭn-Rosenblum equation was G.W. Stewart in a series of papers between 1971 and 1973 (see [Ste71], [Ste72], [Ste73b]). He introduced a special implicit measure $\delta(A,D)$ for the separation of the spectra of $A$ and $D$, which is defined *e.g.* if one of them is bounded and, in the self-adjoint case, reduces to the usual distance. Assuming $\sqrt{\|B\|\,\|C\|} < \delta(A,D)/2$, Stewart proved the existence of a bounded solution of the associated Riccati equation, which guarantees the existence of an invariant subspace of a closed linear operator (see below). Motivated by applications to two-channel Hamiltonians from elementary particle physics, A.K. Motovilov applied the fixed point method to self-adjoint diagonally dominant block operator matrices in 1991/1995 (see [Mot91], [Mot95]); he proved existence and uniqueness of solutions of Riccati equations if dist $\big(\sigma(A), \sigma(D)\big) > 0$ and the Hilbert-Schmidt norm of $B$ satisfies $\|B\|_2 < \text{dist}\,\big(\sigma(A), \sigma(D)\big)/2$. This approach was further advanced, and more general conditions were found, for the non-self-adjoint case in [ALT01] with bounded $B$, $C$, $D$, in [AMM03] with bounded $B$, $C$, and in [AM05] for the case of one normal diagonal entry. The fixed point method also applies if the spectra of $A$ and $D$ are not disjoint: In a series of papers, R. Mennicken, A.K. Motovilov, and V. Hardt (see [MM98], [MM99], [HMM02], [HMM03]) used it for upper dominant block operator matrices for which $\sigma(D)$ is partly or entirely embedded in the continuous spectrum $\sigma_c(A)$ of $A$; they assumed that the second Schur complement admits an analytic continuation under the cuts along the branches of the absolutely continuous spectrum of $A$, which is ensured by conditions on $B$. In numerical linear algebra, iterative schemes for solving Riccati equations were used by K. Veselič and E. Kovač Striko in [KSV01] in order to establish an algorithm for block diagonalizing non-self-adjoint matrices (see below).

Riccati equations play an important role in mathematical systems theory (see the monographs [LR95, Chapter IV] by L. Rodman and P. Lancaster, [CZ95, Chapter 6] by R. Curtain and H. Zwart and the bibliographies therein). They arise *e.g.* in linear quadratic (LQ) optimal control on an infinite time interval and have been the subject of intense research since Kalman's seminal paper from 1960 (see [Kal60]). There, and in other areas like computational physics and chemistry (see [Ben00]), the operator coefficients in (IV) have the special properties that $D = -A^*$ and $B$, $C$ are non-negative; the corresponding block operator matrices are called Hamiltonians. In systems theory results on existence and uniqueness of non-negative Hermitian solutions of Riccati equations are sought as well as iterative schemes to approximate solutions for infinite time intervals by solutions for finite time intervals (see *e.g.* [Mår71], [CZ95], [KSV01]). An idea of the vast literature on Riccati equations for Hamiltonians with bounded coefficients is given in [LR95]; for unbounded coefficients, important contributions are due to A. Pritchard and D. Salamon (see [PS84], [PS87]), and to I. Lasiecka and R. Triggiani (see [LT91], [LT00b], [LT00a]) who gave applications to linear partial differential equations with boundary and point controls. Although some of the methods presented here apply to Riccati equations from systems theory (see [LRT97], [LRvdR02], Remark 2.7.26, and Corollary 2.9.23), this wide area is out of the scope of this book.

2.4. *Invariant graph subspaces and block diagonalization.* Invariant subspaces of matrices and linear operators are a key tool in many areas of mathematics and its applications (see the monograph [GLR86] by I.C. Gohberg, P. Lancaster, and L. Rodman for an exhaustive account). The existence of solutions to the Riccati equations (IV) implies that the subspaces

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x \\ K_1 x \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} K_2 y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\} \tag{V}$$

are invariant for the block operator matrix $\mathcal{A}$ formed from their coefficients. As a consequence, the operator $\mathcal{A}$ formally admits the block diagonalization

$$\begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix} = \begin{pmatrix} A + BK_1 & 0 \\ 0 & D + CK_2 \end{pmatrix}. \tag{VI}$$

Vice versa, if a block operator matrix $\mathcal{A}$ possesses invariant subspaces of the form (V), then the operators $K_1$, $K_2$ therein are solutions of the Riccati equations (IV). Hence the existence of an invariant graph subspace of a block operator matrix is equivalent to the existence of a solution of a corresponding Riccati equation.

The operators $K_1$, $K_2$ in (V) are called angular operators since they provide a measure for the perturbation of the invariant subspaces $\mathcal{H}_1 \oplus \{0\}$, $\{0\} \oplus \mathcal{H}_2$ of the block diagonal operator diag $(A, D)$ if $B$ and $C$ are turned on. As an instructive example, we consider a real $2 \times 2$ matrix $\mathcal{A}$ (see [Hal69], [KMM05]). If $\mathcal{A}$ has two different real eigenvalues $\lambda_1$, $\lambda_2$ with eigenvectors $(x_1 \ y_1)^{\mathrm{t}}$, $(x_2 \ y_2)^{\mathrm{t}}$, say $x_1 \neq 0$, then *e.g.*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x_1 \\ k_1 x_1 \end{pmatrix} : x_1 \in \mathcal{H}_1 \right\}, \quad k_1 = \frac{y_1}{x_1} =: \tan\theta;$$

here $\theta \in [0, \pi/2)$ is the angle between the axis $\mathbb{R} \oplus \{0\}$ and the eigenspace $\mathcal{L}_1$. The orthogonal projection $P$ of $\mathbb{C}^2$ onto $\mathcal{L}_1$ is given by

$$P = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{VII}$$

If $P_1$ is the orthogonal projection of $\mathbb{C}^2$ onto $\mathbb{C} \oplus \{0\}$, then it is not difficult to check that $\|P - P_1\| = \sin\theta$. If $\mathcal{A}$ is a bounded block operator matrix that can be block diagonalized and $\mathcal{L}_1$ is an invariant subspace as in (V), then the orthogonal projection $P$ of $\mathcal{H}_1 \oplus \mathcal{H}_2$ on $\mathcal{L}_1$ is given by

$$P = \begin{pmatrix} (I + K_1^* K_1)^{-1} & (I + K_1^* K_1)^{-1} K_1^* \\ K_1 (I + K_1^* K_1)^{-1} & K_1 (I + K_1^* K_1)^{-1} K_1^* \end{pmatrix}.$$

Comparing with formula (VII), we recognize the formal correspondence

$$I + K_1^* K_1 \;\longleftrightarrow\; \frac{1}{\cos^2\theta} = 1 + \tan^2\theta \quad \text{or} \quad \sqrt{K_1^* K_1} \;\longleftrightarrow\; \tan\theta,$$

which is the identity $k_1 = \tan\theta$ in the matrix case. Using the notion of an operator angle $\Theta$ of a pair of subspaces (see *e.g.* [KMM03a, Section 2]), one arrives at the rigorous identities $\sqrt{K_1^* K_1} = \tan\Theta$ and $\|P - P_1\| = \sin\Theta$ where $P_1$ is the orthogonal projection of $\mathcal{H}_1 \oplus \mathcal{H}_2$ on $\mathcal{H}_1 \oplus \{0\}$.

Even in the case of matrices or bounded linear operators, there are no simple answers neither to the problem of existence of invariant graph subspaces nor to the problem of existence of solutions of Riccati equations; in the unbounded case, additional problems with domains and closures arise. However, the equivalence of the problems widens the range of methods available for their solution. Besides the fixed point methods for Riccati equations described before, we distinguish four main directions:

For self-adjoint operators $\mathcal{A}$, a geometric approach was initiated by works of C. Davis and W.M. Kahan in the 1960ies (see [Dav63], [Dav65], [DK70]); they studied the perturbation of spectral subspaces of a self-adjoint operator $\mathcal{A}_0$ the spectrum of which has two disjoint components. Note that this induces a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ in which $\mathcal{A}_0$ is block diagonal, say $\mathcal{A}_0 = \operatorname{diag}(A, D)$ with $\sigma(A) \cap \sigma(D) = \emptyset$. Their main results are four different types of theorems, called $\sin\theta$ theorem, $\tan\theta$ theorem, $\sin 2\theta$ theorem, and $\tan 2\theta$ theorem, which give the best possible bound on the angle between the perturbed and unperturbed spectral subspaces.

Independently, and only in the 1990ies, V.M. Adamjan and H. Langer developed a different analytic approach in [AL95] for self-adjoint and $\mathcal{J}$-self-adjoint operators of the form

$$\mathcal{A} = \begin{pmatrix} A & B \\ \pm B^* & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ \pm B^* & 0 \end{pmatrix} = \mathcal{A}_0 + \mathcal{V} \tag{VIII}$$

with bounded off-diagonal perturbation $\mathcal{V}$. Under the stronger assumption $\sigma(D) < \sigma(A)$, but without bounds on the norm of $\mathcal{V}$ in the self-adjoint case, they proved that the interval $\big(\max\sigma(D), \min\sigma(A)\big)$ remains free of spectrum for the perturbed operator $\mathcal{A} = \mathcal{A}_0 + \mathcal{V}$ and that the spectral subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to the intervals $\big[\min\sigma(A), \infty\big)$ and $\big(-\infty, \max\sigma(D)\big]$ admit angular operator representations (V) with a uniform contraction $K_1$ and $K_2 = -K_1^*$ (the latter being a consequence of the orthogonality of $\mathcal{L}_1$ and $\mathcal{L}_2$); their method is based on analytic estimates of integrals over the inverse of the Schur complement.

In 2001 a novel approach involving indefinite inner products was established in [LT01] for block operator matrices (VIII) with non-self-adjoint $\mathcal{A}_0$ and self-adjoint $\mathcal{V}$ (*i.e.* $C = B^*$). It is based on a theorem on accretive linear operators in Krein spaces which applies if we assume that $\operatorname{Re} W(D) \leq 0 \leq \operatorname{Re} W(A)$. In fact, if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is equipped with the indefinite inner product $[\cdot,\cdot] = (\mathcal{J}\cdot,\cdot)$ with $\mathcal{J} = \operatorname{diag}(I, -I)$, then

$$\begin{aligned} \operatorname{Re}\left[\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right] &= \operatorname{Re}\left(\begin{pmatrix} A & B \\ -B^* & -D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= \operatorname{Re}(Ax, x) - \operatorname{Re}(Dx, x) \;\geq\; 0 \end{aligned} \tag{IX}$$

for $(x\ y)^{\mathrm{t}} \in \mathcal{D}(\mathcal{A})$. This theorem yields the existence of invariant subspaces of $\mathcal{A}$ that are maximal non-positive and maximal non-negative with respect to $[\cdot,\cdot]$, provided $\mathcal{A}$ is exponentially dichotomous (see the next subsection). As a consequence of the definiteness of these invariant subspaces, we obtain angular operator representations (V) with contractions $K_1$, $K_2$. This approach does not only furnish a new and more elegant proof for the self-adjoint case treated in [AL95], it also covers non-self-adjoint diagonally dominant and off-diagonally dominant block operator matrices. Note that, for the latter, the off-diagonal part $\mathcal{V}$ can no longer be regarded as a perturbation of the diagonal part $\mathcal{A}_0$ and so, even in the self-adjoint case, none of the previous results applies.

Finally, in parallel, a fourth method was developed in [LMMT01] which relies on the factorization theorems by Markus and Matsaev used for the Schur complements. It applies to bounded linear operators $\mathcal{A}$ and decompositions $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that the closure of the quadratic numerical range consists of two disjoint components. In this case, *e.g.* the angular operator $K_1$ and the operator $Z_1$ in the linear factor of $S_1(\lambda) = M_1(\lambda)(Z_1 - \lambda)$ are related by the formulae

$$Z_1 = A + BK_1, \quad K_1 = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C (Z_1 - \lambda)^{-1} \mathrm{d}\lambda;$$

here $\Gamma_1$ is a Cauchy contour separating the two components of the quadratic numerical range. Similarly, we have $Z_2 = D + CK_2$ and a corresponding integral formula for $K_2$. This shows that, indeed, the operators $Z_1$, $Z_2$ in the factorizations (III) of the Schur complements are the diagonal entries in the block diagonalization (VI).

Numerous papers were published following one or two of the previous approaches. The analytic approach by Adamjan and Langer was further pursued for unbounded upper dominant self-adjoint block operator matrices with $\max \sigma(D) \leq \min \sigma(A)$ in [ALMS96] and, for the first time, for non-self-adjoint $A$ and $D$ with $\operatorname{Re} W(D) \le 0 \le \operatorname{Re} W(A)$, bounded $D$, and $C = B^*$ in [LT98]. The method of factorizing the Schur complements was first applied in [MS96] to allow for a certain overlapping of the spectra of $A$ and $D$. The geometric approach of Davis and Kahan was further elaborated in a series of papers by A.K. Motovilov *et al.* (see [KMM03b] for general bounded $\mathcal{V}$; [KMM03a], [KMM07] for bounded off-diagonal $\mathcal{V}$; [KMM04] for bounded off-diagonal $\mathcal{V}$ and the case $\max \sigma(D) \le \min \sigma(A)$; [KMM05] for off-diagonal $\mathcal{V}$ and the case that $A$ is bounded and $\sigma(A)$ lies in a finite gap of $\sigma(D)$; [AMS07] for bounded off-diagonal $\mathcal{V}$ and [MS06] for unbounded off-diagonal $\mathcal{V}$ and, in both cases, $\sigma(A) \cap \operatorname{conv} \sigma(D) = \emptyset$). As in [KMM07] and [AMM03], optimal bounds on the norm of $\mathcal{V}$ guaranteeing that the perturbed spectrum remains separated and on the angle between the perturbed and the unperturbed spectral subspaces are given.

2.5. *Dichotomous block operator matrices.* A linear operator is called dichotomous if its spectrum does not intersect the imaginary axis $\mathrm{i}\mathbb{R}$; in this case, the spectrum of $\mathcal{A}$ splits into two parts $\sigma_1 \subset \mathbb{C}_+$, $\sigma_2 \subset \mathbb{C}_-$ in the open right and left half plane, respectively. The notion of dichotomous operators is closely related to the notion of dichotomy for differential equations. In the most classical case, it means that the solution of a Sturm-Liouville equation on $L_2(\mathbb{R})$ is the sum of two solutions from $L_2(0,\infty)$ and $L_2(-\infty,0)$, respectively. For evolution equations $u'(t) = \mathcal{A}u(t)$, $t \in [0,\infty)$, in an abstract Banach or Hilbert space $\mathcal{H}$, the concept of exponential dichotomy was considered by S.G. Kreĭn and Ju.B. Savčenko (see [KS72]); essentially, it means that there exist two invariant subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{A}$ such that $\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_2$ and $\|u(t)\|$ decays (increases, respectively) exponentially for $t \to \infty$ if $u(t) \in \mathcal{L}_2$ ($u(t) \notin \mathcal{L}_2$, respectively). If $\mathcal{A}$ is a bounded dichotomous operator, then $\mathcal{L}_1$, $\mathcal{L}_2$ can be chosen to be the spectral subspaces of $\mathcal{A}$ corresponding to $\sigma_1$, $\sigma_2$, *i.e.* the ranges of the corresponding Riesz projections; if $\mathcal{A}$ is self-adjoint and unbounded, then $\mathcal{L}_1$, $\mathcal{L}_2$ can be chosen to be the ranges of the corresponding spectral projections. If, however, $\mathcal{A}$ is unbounded and not self-adjoint, then both $\sigma_1$ and $\sigma_2$ may be unbounded and the problem of "separating the spectrum at infinity" (see [GGK90, Section XV.3]) arises.

If an unbounded non-self-adjoint block operator matrix $\mathcal{A}$ as in (VIII) is exponentially dichotomous, then it can be transformed into block diagonal form. The spectral inclusion theorem for the quadratic numerical range yields a criterion for dichotomy. If we assume that the numerical ranges of $A$ and $D$ are separated by a strip around $\mathrm{i}\mathbb{R}$, *i.e.* for some $\alpha$, $\delta > 0$

$$W(A) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq \alpha\}, \quad W(D) \subset \{z \in \mathbb{C} : \operatorname{Re} z \leq -\delta\},$$

that the strip $S := \{z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha\}$ contains at least one point of $\rho(A) \cap \rho(D)$, and that certain relative boundedness assumptions are satisfied for the entries of $\mathcal{A}$, then $S \subset \rho(\mathcal{A})$ and hence $\mathcal{A}$ is dichotomous. Note that this implies that the block operator matrix $\mathcal{A}$ is $\mathcal{J}$-accretive (see (IX)). If we require, in addition, that $W(A)$ and $W(D)$ lie in certain sectors of angle less than $\pi$ in the right and left half plane, respectively, then $\mathcal{A}$ is exponentially dichotomous. This follows from a deep theorem proved by H. Bart, I.C. Gohberg, and M.A. Kaashoek (see [BGK86, Theorem 3.1], [GGK90, Theorem XV.3.1]); they studied exponentially dichotomous operators intensively in relation with Wiener-Hopf factorization. Equivalent conditions for the separation of the spectrum at infinity were given by G. Dore and A. Venni in [DV89] in terms of powers of the operator in question. In [RvdM04], [vdMR05], A.C.M. Ran and C. van der Mee considered additive and multiplicative perturbations of exponentially dichotomous operators; they used methods different from those presented here, *e.g.* the Bochner-Phillips theorem. A survey of this area and of applications of exponentially dichotomous operators, *e.g.* to transport equations, diffusion equations of indefinite Sturm-Liouville type, noncausal infinite-dimensional linear continuous-time systems, and functional differential equations of mixed type are given in the recent monograph [vdM08] by C. van der Mee.

2.6. *Variational principles and eigenvalue estimates.* The variational characterization of eigenvalues goes back well into the 19th century, to H. Weber (see [Web69]) and Lord Rayleigh (see [Ray26]). If $\mathcal{A}$ is a self-adjoint operator that is semi-bounded, say from below, then the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ of $\mathcal{A}$ below its essential spectrum can be characterized by means of the classical min-max principle

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathcal{D}(\mathcal{A}) \\ \dim \mathcal{L} = n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} p(x), \quad p(x) := \frac{(\mathcal{A}x, x)}{\|x\|^2}, \tag{X}$$

(see *e.g.* [RS78, Chapter XIII], [WS72], [Gou57], [Ste70]). Here $p$ is the so-called Rayleigh functional defining the numerical range of $\mathcal{A}$. Besides the min-max principle, which is based on the inequalities of H. Poincaré (see [Poi90]), there also exists a max-min characterization which relies on the inequalities of H. Weyl (see [Wey12]). Min-max and max-min principles are effective tools in the qualitative and quantitative analysis of eigenvalues of self-adjoint operators for several reasons. They do not require any knowledge about eigenvectors and can be used for comparing eigenvalues of operators, deriving eigenvalue estimates, locating the bottom of the essential spectrum (or showing it is empty), proving the existence of eigenvalues below the essential spectrum, and for numerical approximations of eigenvalues; corresponding algorithms have been used in countless applications from physics and engineering sciences for decades, *e.g.* in elasticity theory for calculating buckling loads of beams or plates (see *e.g.* [Mik64]).

Due to the convexity of the numerical range, the classical variational principles only apply to eigenvalues to the left (or to the right) of the essential spectrum, but not to eigenvalues in gaps of the essential spectrum. This excludes eigenvalues of some important operators from mathematical physics like Dirac operators, Klein-Gordon operators, and Schrödinger operators with periodic potentials. The first abstract min-max principles for block operator matrices with spectral gap were proved by M. Griesemer and H. Siedentop in 1999 and generalized in 2000 jointly with R.T. Lewis (see [GS99], [GLS99]), followed by work of M.J. Esteban, J. Dolbeault, and E. Séré beginning in 2000 (see [DES00b]), [DES00c], [DES06]). The main motivation of these authors was to characterize the eigenvalues of Dirac operators with Coulomb potential, as suggested in earlier work of J. Talman and of S.N. Datta and G. Deviah (see below). While the assumptions and methods of proof are different, the common idea in these papers is to use the given decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and to impose the dimension restriction only in one component, *e.g.* in the bounded case

$$\lambda_n = \min_{\substack{\mathcal{L}_1 \subset \mathcal{H}_1 \\ \dim \mathcal{L}_1 = n}} \max_{\substack{x \in \mathcal{L}_1 \oplus \mathcal{H}_2 \\ x \neq 0}} p(x). \tag{XI}$$

Much earlier, beginning in the 1950ies, min-max principles were proved for operator functions depending non-linearly on the spectral parameter. In 1955 R. Duffin was the first to consider the case of self-adjoint quadratic operator polynomials $T(\lambda)$ (see [Duf55] and the later work [Bar74] of E.M. Barston); self-adjoint continuously differentiable functions of matrices and of bounded linear operators defined on some real interval $I$ were studied in 1964 by E.H. Rogers (see [Rog64], [Rog68]) and by B. Werner in 1971 (see [Wer71]), respectively. Here a generalized Rayleigh functional $p$ was introduced which, *e.g.* if $T(\cdot)$ is strictly monotonically increasing, is defined to be the unique zero $p(x)$ of $\big(T(\lambda)x, x\big)$ (or $\pm\infty$ if no zero exists). This definition is closely related to the numerical range of the operator function $T$, which is given by $W(T) := \big\{\lambda \in I : \big(T(\lambda)x, x\big) = 0 \text{ for some } x \in \mathcal{H}, x \neq 0\big\}$. In the particular case $T(\lambda) = \mathcal{A} - \lambda$, the unique zero of $\big(T(\lambda)x, x\big) = 0$ is just the classical Rayleigh functional $p(x)$ (see (X)). Under much weaker assumptions on the operator functions, these min-max principles were generalized by P. Binding, H. Langer, and D. Eschwé in [BEL00]) for the case of bounded values $T(\lambda)$, and by D. Eschwé, and M. Langer in [EL04] for the case of unbounded values $T(\lambda)$. In both papers, the variational principles were applied to the Schur complements to characterize eigenvalues of self-adjoint and even skew-self-adjoint block operator matrices.

A new type of variational principles for eigenvalues in spectral gaps appeared in 2002 (see [LLT02], [KLT04]). They apply to block operator matrices having real quadratic numerical range $W^2(\mathcal{A})$, *i.e.* to self-adjoint and certain skew-self-adjoint block operator matrices. Here the role of the classical Rayleigh functional is played by the functionals $\lambda_\pm$ induced by the quadratic numerical range; they are defined as the zeroes $\lambda_\pm\binom{x}{y}$ of the quadratic polynomial $\det(\mathcal{A}_{x,y}-\lambda)$ (*i.e.* the eigenvalues of the matrix $\mathcal{A}_{x,y}$ given by (II)). The proof of these novel min-max and max-min principles uses the variational principle of [EL04] and the inclusion of the numerical range of the Schur complement in the quadratic numerical range. As a corollary, we obtain the min-max principle (XI) with the classical Rayleigh functional $p$. In the off-diagonally dominant case, the results of [KLT04] are restricted to bounded diagonal entries; a generalization to one relatively bounded diagonal entry was given in [Win05] (see also [Win08]).

The problem of eigenvalue accumulation in gaps of the essential spectrum of self-adjoint block operator matrices was investigated also in [AMS98]; some of the results therein follow from more general considerations of V.A. Derkach and M.M. Malamud on self-adjoint extensions of symmetric operators with spectral gaps (see [DM91]).

2.7. *Motivation by applications.* Many physical systems are described by systems of partial or ordinary differential equations or linearizations thereof. The corresponding spectral problems tend to be challenging; profound physical intuition and advanced techniques from the analysis of differential equations are common ways to address them. The theory of block operator matrices opens up a new line of attack.

*Magnetohydrodynamics and fluid mechanics.* The study of block operator matrices occurring in magnetohydrodynamics and fluid mechanics was initiated by several papers of G. Grubb, G. Geymonat (see [GG74], [GG77], [GG79]) and of J. Descloux, G. Geymonat (see [DG79], [DG80]). Using pseudo-differential calculus, they developed methods to determine the essential spectrum of so-called Douglis-Nirenberg elliptic systems. It seems that these authors were the first to observe that, unlike regular differential operators, regular *matrix* differential operators may have non-empty essential spectrum. Applications of the results were given *e.g.* to the linearized Navier-Stokes operator from fluid mechanics.

In magnetohydrodynamics, knowledge of the essential spectrum, especially of its so-called Alfvén range part, may be used to heat up the plasma inexpensively (see [Lif89, Section 7.9]). A variety of singular and regular spectral problems from magnetohydrodynamics were studied by T. Kako (see [Kak84], [Kak85], [Kak87], [Kak88a], [Kak88b], [KD91], [Kak94]) and by G. Raĭkov by means of variational techniques (see [Raĭ85], [Raĭ86a], [Raĭ86b], [Raĭ87], [Raĭ88], [Raĭ90], [Raĭ91], [Raĭ97]); G. Raĭkov also investigated the discrete eigenvalues and proved asymptotic formulae for them.

The methods developed in [ALMS94] for upper dominant block operator matrices were employed to determine the essential spectrum for several problems in magnetohydrodynamics, fluid mechanics, and astrophysics: regular two-dimensional problems from magnetohydrodynamics were studied by M. Faierman, A. Yu. Konstantinov, H. Langer, R. Mennicken, and M. Möller (see [FLMM94], [FMM95], [LM96], [Kon02]); singular problems were considered in [FMM99], [FMM00], and by S.N. Naboko *et al.* in [HMN99], [MNT02], [KN02], [KN03]. In the more involved singular case considered in [DG79], Descloux and Geymonat's conjecture on the essential spectrum is still waiting to be proved; by now, it was only confirmed for a simpler model problem in [FMM04]. The result for the essential spectrum of the linearized Navier-Stokes operator obtained in [GG79] was reproved in [FFMM00]. The methods of [ALMS94] apply also to problems from the stability analysis of stellar oscillations, *e.g.* to a model arising in the Cowling approximation, which was considered by H. Beyer directly by Sturm-Liouville techniques (see [Bey00]), or to differentially rotating stars (see [FLMM99]).

*Quantum mechanics.* Another field where block operator matrices naturally occur is quantum mechanics, the most prominent example being the Dirac operator (see [Tha92]):

$$\mathbf{H_\Phi} := \begin{pmatrix} (mc^2 + e\Phi)\, I & c\,\vec{\sigma}\cdot\left(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\right) \\ c\,\vec{\sigma}\cdot\left(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\right) & (-mc^2 + e\Phi)\, I \end{pmatrix}.$$

Here the problem of block diagonalization amounts to a decoupling of positive- and negative-energy states (electrons and positrons). For the free Dirac operator in $\mathbb{R}^3$ (*i.e.* $\vec{A} = 0$, $\Phi = 0$), an exact unitary transformation into block diagonal form was found by L.L. Foldy and S.A. Wouthuysen in 1950 (see [FW50] and the review articles [DV70], [CM95]). For non-vanishing electric potential $\Phi$, the Foldy-Wouthuysen method suggests a sequence of unitary transformations eliminating the lowest order off-diagonal term in $1/c$ in each step. However, the resulting series expansion in $1/c$ for the successively transformed Dirac operator is ill-behaved (see [GNP89], [RW04a]). In 1974, another iterative diagonalization scheme was proposed by M. Douglas and N.M. Kroll who used an expansion in terms of orders in the potential $\Phi$ rather than in $1/c$ (see [DK74b]); its efficiency for quantum chemical implementations was first noticed by B.A. Hess in [Heß86] and investigated in detail in a recent series of papers by M. Reiher and A. Wolf (see [RW04a], [RW04b], [WR06a], [WR06b]). Unlike the Foldy-Wouthuysen approximation, this method yields a well-behaved series with a limiting block diagonal operator having the same spectrum as the original Dirac operator (see [RW04a]). For Coulomb potentials up to atomic number 51, convergence of this series in norm resolvent sense was proved recently by H. Siedentop and E. Stockmeyer in [SS06]; numerically, Reiher and Wolf have tested convergence for the ground state energies of one-electron ions over the whole periodic table in [RW04b]. In 1988, B. Thaller generalized the Foldy-Wouthuysen transformation for abstract supersymmetric Dirac operators, which include the Dirac operator with vector potential $\vec{A}$, but not with electric potential (see [Tha88], [Tha91], and the monograph [Tha92]).

The idea of achieving a block diagonalization for various Hamiltonians in quantum mechanics by means of invariant graph subspaces may be traced back to 1954 to S. Ôkubo (see [Ôku54] and also the survey paper [SS01] for an introduction to Ôkubo's method). His approach includes free Dirac operators as a special case, but it also has applications in elementary particle physics (see e.g. [GM81], [KS93], and [SS01]); there the two components of the Hilbert space correspond, for example, to a space of external (*e.g.* hadronic) degrees of freedom and a space of internal (*e.g.* quark) degrees of freedom (see [DHM76], [Mot95]). In a mathematically rigorous way, Ôkubo's idea was used by V.A. Malyshev and R.A. Minlos for the construction of invariant subspaces for a class of self-adjoint block operator matrices in statistical physics, so-called clustering operators (see [MM79], [MM82]); their assumptions amount to separated spectra of the diagonal entries and sufficiently small off-diagonal terms.

The first exact Foldy-Wouthuysen transformation for Dirac operators with electric potentials was proved in 2001 in [LT01]. This result is a consequence of the abstract theorem on block diagonalization described above; at the moment, it is restricted to bounded potentials, but we conjecture that the method continues to be applicable as long as the potential does not mix the positive and negative part of the spectrum too badly. This conjecture is supported by the recent paper [Cor04] of H.O. Cordes. Independently of [LT01], in 2004, he proved the existence of an exact Foldy-

Wouthuysen transformation for Dirac operators with unbounded smooth electromagnetic fields; in [Cor04], the latter is ensured by the assumption that all derivatives of order $k$ decay like $(1+|x|)^{-k-1}$ (see also [Nen76]). Such fields do not alter the essential spectrum, but may produce a finite mixing of eigenvalues emerging from the left and the right end-point of the essential spectrum. Cordes' method relies on his earlier works on pseudo-differential calculus and its application to Dirac operators, begun already in the 1980ies (see [Cor79], [Cor83b], [Cor83a], [Cor95], [Cor00], [Cor01]).

Variational principles for Dirac operators of the form (XI) were first suggested in the 1980ies by J.D. Talman (see [Tal86]) and by S.N. Datta and G. Deviah (see [DD88]). The main difficulty, compared to Schrödinger operators, is that the Dirac operator is not semi-bounded and eigenvalues appear in a gap of the essential spectrum. Remarkably, Talman already used the Schur complement implicitly for his heuristic arguments (see [Tal86, (4)]). The first rigorous min-max result for Dirac operators with Coulomb potential was considered in 1997 by M. Esteban *et al.* (see [ES97] and also [DES00b]). Later the abstract min-max principles by H. Siedentop *et al.* in [GS99], [GLS99] and by M. Esteban *et al.* in [DES06] were applied to Dirac operators with Coulomb-like potentials. The variational principles in [KLT04] were applied to Dirac operators with bounded potentials and to Dirac operators on Riemannian spin manifolds with warped product metric. Recently, M. Winklmeier used block operator methods to establish variational principles for eigenvalues of the angular part of the Dirac operator in curved spacetime (see the PhD thesis [Win05] and [Win08]). The resulting eigenvalue estimates are the first analytic results, preceded only by numerical calculations of K.G. Suffern, E.D. Fackerell, C.M. Cosgrove and of S.K. Chakrabarti in the 1980ies (see [SFC83], [Cha84]).

*Other areas.* There are many other applications that we do not touch in this book. They include linear evolution problems that can be written formally as first order Cauchy problems

$$\dot{u}(t) = \mathcal{A}\, u(t), \quad u(0) = u_0, \tag{XII}$$

with a block operator matrix $\mathcal{A}$ in a product space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ and a function $u : \mathbb{R}^+ \to \mathcal{H}$ (see *e.g.* [Nag89]). Often these problems arise as linearizations of second order Cauchy problems; examples include the wave equation (see *e.g.* [Gol85]) and the Klein-Gordon equation from quantum mechanics (see *e.g.* [Lun73]) for which

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ B & D \end{pmatrix}, \quad B = c^2\big(-\mathrm{i}\hbar\nabla - \tfrac{e}{c}\vec{A}\big)^2 + m^2c^4 - e^2\Phi^2, \quad D = 2e\Phi;$$

other linearizations lead to different coefficients $\mathcal{A}$ in (XII) (see *e.g.* [Ves91]). Here the crucial problem is to show that the operator $\mathcal{A}$ is the generator of a semi-group. For wave equations, the operator matrix approach initiated by Nagel was applied recently by D. Mugnolo (see [Mug06a], [Mug06c], [Mug06b]). Other classes of block operator matrices, such as one-sided coupled operator matrices or polynomial operator matrices, were studied intensively by K.-J. Engel *et al.* in numerous papers (see [Eng89a], [Eng89b], [EN90], [Eng90], [Eng91a], [Eng91b], [Eng93], [Eng96], [Eng98], [Eng99], [BE04]); applications range from epidemiology to elasticity theory and reaction-diffusion systems. Operator matrices with non-diagonal domain in the sense of Nagel and applications to differential equations with delay were considered in [BBD$^+$05]. For abstract operator matrices of Klein-Gordon type, the block operator matrix approach presented in this book and indefinite inner product methods were applied by various authors, including K. Veselič, B. Najman, and P. Jonas (see [Ves91], [Naj83], [Jon93], [Jon00], [LNT06], [LNT08]); variational principles for the Klein-Gordon equation, even in the presence of complex eigenvalues, were derived in [LT06]. Along similar lines, Cauchy problems arising in hydrodynamics describing small oscillations of a fluid in a partially filled container were studied in [AHKM03], [KMPT04]).

**3. Outline of contents.** The three chapters of this book are organised in the following way:

Chapter 1 deals with *bounded block operator matrices.* In Section 1.1 the quadratic numerical range $W^2(\mathcal{A})$ of a bounded block operator matrix $\mathcal{A}$ is introduced and some elementary properties are proved. In Section 1.2 we study the quadratic numerical range of several special classes of block operator matrices. Section 1.3 contains the most important property of the quadratic numerical range, the spectral inclusion. In Section 1.4 we establish an estimate of the resolvent in terms of the quadratic numerical range. In Section 1.5 the corners of the quadratic numerical range are studied. In Section 1.6 we analyse the Schur complements of the block operator matrix and the relation of their numerical ranges to the quadratic numerical range. We show that if the closure of the quadratic numerical range consists of two disjoint components, then the Schur complements admit factorizations. This result is used in Section 1.7 to prove one of the main results of this chapter, the theorem on block diagonalization (see Theorem 1.7.1). A consequence of this theorem is the existence of bounded solutions of the corresponding Riccati equations. In Section 1.8 so-called spectral supporting subspaces $\mathcal{H}_1^{\Delta}$ of self-adjoint block operator matrices are studied; they are characterized by the property that the spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ is the graph of a linear operator defined on $\mathcal{H}_1^\Delta$. In Section 1.9 variational principles based on the functionals $\lambda_\pm$ defining the quadratic numerical range are established; they apply to eigenvalues in gaps of the essential spectrum of bounded linear operators with real quadratic numerical range. Section 1.10 deals with so-called $\mathcal{J}$-self-adjoint block operator matrices (*i.e.* $A = A^*$, $D = D^*$, $C = -B^*$). The functionals $\lambda_\pm$ are used to classify the spectral points of $\mathcal{A}$ and to identify an interval $[\nu, \mu]$ outside of which the spectrum of $\mathcal{A}$ is of definite type; the latter implies that $\mathcal{A}$ possesses a local spectral function on $\mathbb{R} \setminus [\nu, \mu]$. In addition, the results on spectral supporting subspaces are extended to the case $C = -B^*$. In Section 1.11 the quadratic numerical range is generalized to $n \times n$ block operator matrices; all results of Sections 1.1 to 1.4 carry over to this so-called block numerical range. Moreover, we prove that a refinement of the decomposition of the Hilbert space leads to an inclusion of the corresponding block numerical ranges. Together with the spectral inclusion property, this yields a successively improved localization of the spectrum. In Section 1.12 we show that the block numerical range of the companion matrix of an operator polynomial contains its numerical range. Finally, in Section 1.13, Gershgorin's theorem for $n \times n$ block operator matrices is presented and compared with the spectral inclusion by means of the block numerical range.

Chapter 2 focuses on *unbounded block operator matrices.* Section 2.1 contains some basic properties of closed linear operators; in particular, the notions of relative boundedness and relative compactness are presented. In Section 2.2 we establish criteria for closedness and closability of block operator matrices. To this end, three classes of block operator matrices are distinguished, depending on the positions of the "strongest" operators in each column (*i.e.* the entries with smallest domains): diagonally dominant, off-diagonally dominant, and upper dominant block operator matrices. In Section 2.3 and Section 2.4 we investigate the spectrum and the essential spectrum, respectively, of block operator matrices by means of Schur complements and quadratic complements. In Section 2.5 we introduce the quadratic numerical range for unbounded block operator matrices and prove the spectral inclusion theorem for diagonally dominant and off-diagonally dominant block operator matrices. In Section 2.6 some properties of symmetric and $\mathcal{J}$-symmetric block operator matrices are studied; in particular, a classification of eigenvalues in terms of the quadratic numerical range is given. Section 2.7 contains some of the key results of Chapter 2 (see Theorems 2.7.7, 2.7.21, and 2.7.24); they concern the existence of invariant graph subspaces and of solutions of Riccati equations for dichotomous block operator matrices. These theorems apply to essentially self-adjoint block operator matrices and, in the non-self-adjoint case, to diagonally dominant and off-diagonally dominant block operator matrices with certain relative boundedness assumptions between the entries (ensuring exponential dichotomy). In Section 2.8 we exploit the theorems of Section 2.7 further to obtain results on block diagonalization as well as on half range completeness and half range basis properties of eigenfunctions and associated functions. In Section 2.9 we reconsider the existence problem for solutions of Riccati equations by means of fixed point methods, which also provide uniqueness of the solution and a convergent iteration scheme. In Section 2.10 we derive variational principles for eigenvalues in gaps of the essential spectrum for self-adjoint and $\mathcal{J}$-self-adjoint block operator matrices. In Section 2.11 we use the variational principles to derive two-sided eigenvalue estimates. Many of the results of Chapter 2 are illustrated by matrix differential operators.

Chapter 3 concentrates on a selection of *applications in mathematical physics* to which the results of Chapter 2 are applied. In Section 3.1 we consider a stability problem for small oscillations of a magnetized gravitating plane equilibrium layer of a hot compressible ideal plasma, which is described by an upper dominant essentially self-adjoint block operator matrix. In particular, we derive a formula for the essential spectrum, which consists of two intervals called slow magnetosonic and Alfvén spectrum, and we give estimates for the eigenvalues if there is a gap between these two intervals. In Section 3.2 we study the stability of the two-dimensional Ekman boundary layer flow which is produced in a rotating tank with small inflow; the corresponding block operator matrix is diagonally dominant and non-self-adjoint. We calculate the essential spectrum which consists of a curve in the complex plane confined to a semi-strip. Section 3.3 is dedicated to Dirac operators in $\mathbb{R}^3$ and in curved spacetime; in both cases, the corresponding block operator matrices are self-adjoint and off-diagonally dominant. We show that, for the free Dirac operator in $\mathbb{R}^3$, our diagonalization theorem reproduces the well-known Foldy-Wouthuysen transformation. In contrast to the latter, our method also applies to Dirac operators in $\mathbb{R}^3$ with electric potential under some boundedness condition. Finally, we study the angular part of the Dirac equation in the spacetime generated by an electrically charged rotating massive black hole. We use the variational principles of Section 2.10 to derive upper and lower bounds for the eigenvalues and compare them with numerical values from the physics literature.

# Contents

## Chapter 1

# Bounded Block Operator Matrices

A block operator matrix is a matrix the entries of which are linear operators. Every bounded linear operator can be written as a block operator matrix if the space in which it acts is decomposed in two or more components. In this chapter we present methods that allow us to use information on the entries in such a representation to investigate the spectral properties of the given operator. The key tool here is the quadratic numerical range or, more generally, the block numerical range. Our main results include a spectral inclusion theorem, an estimate of the resolvent in terms of the quadratic numerical range, factorization theorems for the Schur complements, and a theorem about angular operator representations of spectral invariant subspaces; the latter implies *e.g.* the existence of solutions of the corresponding Riccati equations and a block diagonalization. Many of the results are also of interest for partitioned matrices.

### 1.1 The quadratic numerical range

The numerical range is an important tool in the spectral analysis of bounded and unbounded linear operators in Hilbert spaces. We begin by collecting some if its useful properties (see *e.g.* [Hal82], [GR97], [Kat95], and [Ber62]).

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a bounded linear operator in $\mathcal{H}$. Then the numerical range of $\mathcal{A}$ is the set

$$W(\mathcal{A}) := \big\{(\mathcal{A}\mathbf{x}, \mathbf{x}) : \mathbf{x} \in S_{\mathcal{H}}\big\}$$

where $S_{\mathcal{H}} := \{\mathbf{x} \in \mathcal{H} : \|\mathbf{x}\| = 1\}$ is the unit sphere in $\mathcal{H}$. By the well-known Toeplitz-Hausdorff theorem, the numerical range is a convex subset of $\mathbb{C}$ and it satisfies the so-called *spectral inclusion property*

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})} \tag{1.1.1}$$

for the point spectrum $\sigma_{\rm p}(\mathcal{A})$ (or set of eigenvalues) and the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$; note that $W(\mathcal{A})$ is closed if $\dim \mathcal{H} < \infty$. Further, the resolvent of $\mathcal{A}$ can be estimated in terms of the distance to the numerical range,

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{1}{\operatorname{dist}\big(\lambda, W(\mathcal{A})\big)}, \qquad \lambda \notin \overline{W(\mathcal{A})}. \tag{1.1.2}$$

If a point $\lambda \in \overline{W(\mathcal{A})}$ is a corner of the numerical range (*i.e.* $W(\mathcal{A})$ lies in a sector with vertex $\lambda$ and angle less than $\pi$), then $\lambda \in \sigma(\mathcal{A})$; if, in addition, $\lambda \in W(\mathcal{A})$, then $\lambda \in \sigma_{\rm p}(\mathcal{A})$. The estimate (1.1.2) implies that if $\lambda \in \sigma_{\rm p}(\mathcal{A})$ is a boundary point of $W(\mathcal{A})$, then there are no associated vectors at $\lambda$.

If the Hilbert space $\mathcal{H}$ is the product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then every bounded linear operator $\mathcal{A} \in L(\mathcal{H})$ has a block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1.3}$$

with bounded linear operators $A \in L(\mathcal{H}_1)$, $B \in L(\mathcal{H}_2, \mathcal{H}_1)$, $C \in L(\mathcal{H}_1, \mathcal{H}_2)$, and $D \in L(\mathcal{H}_2)$. The following generalization of the numerical range of $\mathcal{A}$ takes into account the block structure (1.1.3) of $\mathcal{A}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

**Definition 1.1.1** For $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ we define the $2 \times 2$ matrix

$$\mathcal{A}_{f,g} := \begin{pmatrix} (Af,f) & (Bg,f) \\ (Cf,g) & (Dg,g) \end{pmatrix} \in M_2(\mathbb{C}). \tag{1.1.4}$$

Then the set

$$W^2(\mathcal{A}) \ := \bigcup_{f \in S_{\mathcal{H}_1}, g \in S_{\mathcal{H}_2}} \sigma_{\rm p}(\mathcal{A}_{f,g}) \tag{1.1.5}$$

is called the *quadratic numerical range* of $\mathcal{A}$ (with respect to the block operator matrix representation (1.1.3)).

For two different decompositions of the Hilbert space $\mathcal{H}$, the corresponding quadratic numerical ranges may differ considerably:

**Example 1.1.2** The quadratic numerical ranges of the $4 \times 4$ matrix

$$\mathcal{A}_0 := \begin{pmatrix} -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ -2 & -1 & 0 & -3\mathrm{i} \\ -1 & -2 & 3\mathrm{i} & 0 \end{pmatrix}$$

with respect to the two decompositions $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ and $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}^1$ are shown in Fig. 1.1; the black dots mark the eigenvalues of $\mathcal{A}_0$.

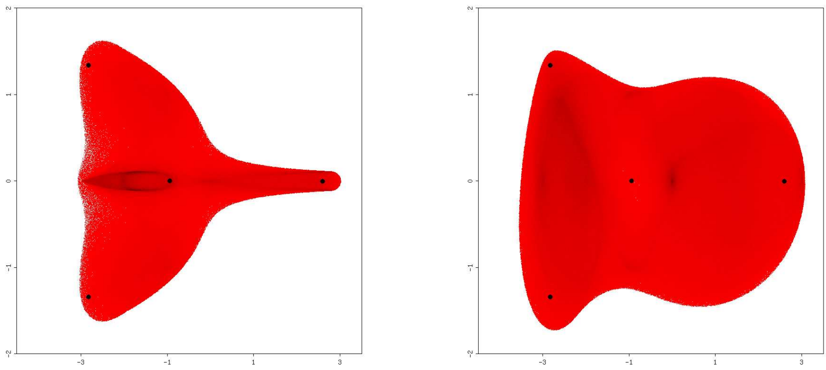

Figure 1.1 Quadratic numerical ranges of $\mathcal{A}_0$ for $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ and $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}^1$.

Sometimes it is more convenient to use an equivalent description of the quadratic numerical range which uses non-zero elements $f$, $g$ that need not have norm one.

**Proposition 1.1.3** *For $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, $f$, $g \neq 0$, we define*

$$\mathcal{A}_{f,g} := \begin{pmatrix} \dfrac{(Af,f)}{\|f\|^2} & \dfrac{(Bg,f)}{\|f\|\,\|g\|} \\ \dfrac{(Cf,g)}{\|f\|\,\|g\|} & \dfrac{(Dg,g)}{\|g\|^2} \end{pmatrix} \in M_2(\mathbb{C}) \tag{1.1.6}$$

*and*

$$\Delta(f,g;\lambda) := \det \begin{pmatrix} (Af,f) - \lambda(f,f) & (Bg,f) \\ (Cf,g) & (Dg,g) - \lambda(g,g) \end{pmatrix}. \tag{1.1.7}$$

*Then*

$$\begin{aligned} W^2(\mathcal{A}) &= \bigcup_{\substack{f\in\mathcal{H}_1, g\in\mathcal{H}_2 \\ f,g\neq 0}} \sigma_{\mathrm{p}}(\mathcal{A}_{f,g}) \\ &= \bigl\{\lambda \in \mathbb{C} : \exists\, f\in\mathcal{H}_1,\ g\in\mathcal{H}_2,\ f,g \neq 0\ \ \det\bigl(\mathcal{A}_{f,g} - \lambda\bigr) = 0\bigr\} \\ &= \bigl\{\lambda \in \mathbb{C} : \exists\, f\in\mathcal{H}_1,\ g\in\mathcal{H}_2,\ f,g \neq 0\ \ \Delta(f,g;\lambda) = 0\bigr\}. \end{aligned}$$

***Proof.*** The claims are immediate if we observe that the definition of $\mathcal{A}_{f,g}$ in (1.1.6) coincides with the one in (1.1.4) if $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, *i.e.* $\|f\| = \|g\| = 1$, and that $\Delta(f,g;\lambda) = \|f\|^2\|g\|^2 \det\bigl(\mathcal{A}_{f,g} - \lambda\bigr)$. □

In the special case that $W^2(\mathcal{A})$ is real, it can also be described by means of the formulae for the roots of the quadratic equation $\det\bigl(\mathcal{A}_{f,g} - \lambda\bigr) = 0$.

In the following, we choose a branch of the square root such that $\sqrt{z} \geq 0$ if $z \geq 0$ and $\operatorname{Im}\sqrt{z} > 0$ if $z < 0$.

**Corollary 1.1.4** *For $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, $f, g \neq 0$, we define*

$$\operatorname{dis}_{\mathcal{A}}(f,g) := \left(\frac{(Af,f)}{\|f\|^2} - \frac{(Dg,g)}{\|g\|^2}\right)^2 + 4\frac{(Bg,f)(Cf,g)}{\|f\|^2\,\|g\|^2}$$

*and, if $\operatorname{dis}_{\mathcal{A}}(f,g) \geq 0$, we set*

$$\lambda_\pm\begin{pmatrix} f \\ g \end{pmatrix} := \frac{1}{2}\left(\frac{(Af,f)}{\|f\|^2} + \frac{(Dg,g)}{\|g\|^2} \pm \sqrt{\left(\frac{(Af,f)}{\|f\|^2} - \frac{(Dg,g)}{\|g\|^2}\right)^2 + 4\frac{(Bg,f)(Cf,g)}{\|f\|^2\,\|g\|^2}}\right).$$

*Further we let*

$$\Lambda_\pm(\mathcal{A}) := \left\{\lambda_\pm\begin{pmatrix} f \\ g \end{pmatrix} : \ f \in \mathcal{H}_1, g \in \mathcal{H}_2, \ f, g \neq 0, \operatorname{dis}_{\mathcal{A}}(f,g) \geq 0\right\}. \quad (1.1.8)$$

*Then $W^2(\mathcal{A}) \subset \mathbb{R}$ if and only if $\operatorname{dis}_{\mathcal{A}}(f,g) \geq 0$ for all $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, $f, g \neq 0$, and in this case*

$$W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \cup \Lambda_+(\mathcal{A}).$$

For convenience, we also use the notation $\lambda_\pm(f,g)$ in the following.

***Proof.*** The claim is immediate from the fact that $\lambda_\pm(f,g)$ are the solutions of the quadratic equation

$$\lambda^2 - \lambda\left(\frac{(Af,f)}{\|f\|^2} + \frac{(Dg,g)}{\|g\|^2}\right) + \frac{(Af,f)(Dg,g)}{\|f\|^2\|g\|^2} - \frac{(Bg,f)(Cf,g)}{\|f\|^2\|g\|^2} = 0, \quad (1.1.9)$$

that is, of $\det\bigl(\mathcal{A}_{f,g} - \lambda\bigr) = 0$. □

Like the numerical range, the quadratic numerical range of a bounded block operator matrix $\mathcal{A}$ is a bounded subset of $\mathbb{C}$,

$$W^2(\mathcal{A}) \subset \bigl\{\lambda \in \mathbb{C} : \ |\lambda| \leq \|\mathcal{A}\|\bigr\},$$

and it is closed if $\dim \mathcal{H} < \infty$. In contrast to the numerical range, it consists of at most two (connected) components. This follows from the fact that the set of all matrices $\mathcal{A}_{f,g}$, $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, is connected and from a continuity argument for the eigenvalues of matrices (see [Kat95, Theorem II.5.14] and [Wag00]). If, for example, $\mathcal{A}$ is upper or lower block triangular, then $W^2(\mathcal{A}) = W(A) \cup W(D)$. Hence the quadratic numerical range is, in general, not convex; the following example shows that even its components need not be so (see Fig. 1.2).

**Example 1.1.5** Consider the $4 \times 4$ matrices

$$\mathcal{A}_1 := \left(\begin{array}{cc|cc} 1 & 0 & 1 & \mathrm{i} \\ 0 & 1 & 0 & 1 \\ \hline \mathrm{i} & 0 & 0 & 0 \\ 1 & \mathrm{i} & 0 & 0 \end{array}\right), \quad \mathcal{A}_2 := \left(\begin{array}{cc|cc} 2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \mathrm{i} & 2 & 3+\mathrm{i} & 1 \\ \hline 1 & 3+\mathrm{i} & -2 & \mathrm{i} \\ 3+\mathrm{i} & 1 & \mathrm{i} & -2 \end{array}\right)$$

with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. Figure 1.2 shows that in both cases the quadratic numerical range consists of two disjoint non-convex components.

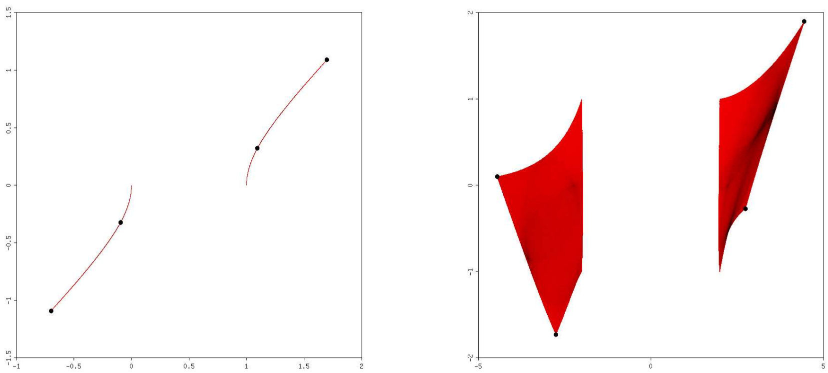

Figure 1.2 Quadratic numerical ranges of $\mathcal{A}_1$ and $\mathcal{A}_2$.

**Remark 1.1.6** The fact that all matrices $\mathcal{A}_{f,g}$, $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, have two different eigenvalues does not imply that $W^2(\mathcal{A})$ consists of two disjoint components.

In fact, there exist self-adjoint block operator matrices $\mathcal{A}$ such that for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ the two eigenvalues $\lambda_+(f,g)$, $\lambda_-(f,g)$ of the matrix $\mathcal{A}_{f,g}$ are different, that is, $\lambda_-(f,g) < \lambda_+(f,g)$, but there exist $f, f' \in S_{\mathcal{H}_1}$, $g, g' \in S_{\mathcal{H}_2}$ such that $\lambda_-(f,g) = \lambda_+(f',g')$. To this end, consider a self-adjoint block operator matrix $\mathcal{A}$ as in (1.1.3) with $\mathcal{H}_1 = \mathcal{H}_2$, $\dim \mathcal{H}_1 \geq 2$, $C = B^*$, and self-adjoint operators $A$, $D$ such that $\min W(A) = \max W(D) = \beta$. Assume further that $\beta$ is a simple eigenvalue of $A$ and $D$ with a common eigenvector $f_0 \in \mathcal{H}_1$, $\|f_0\| = 1$, and $(Bf_0, f_0) \neq 0$. Then it is easy to see that for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$

$$\lambda_+(f,g) - \lambda_-(f,g) = 2\sqrt{\left(\frac{(Af,f)-(Dg,g)}{2}\right)^2 + |(Bg,f)|^2} > 0.$$

On the other hand, if we choose $f_0' \in \mathcal{H}_1$, $g_0' \in \mathcal{H}_1$ so that $\|f_0'\| = \|g_0'\| = 1$, $(Bf_0, f_0') = 0$ and $(B^* f_0, g_0') = 0$, then, by the definition of $\lambda_\pm$ in Corollary 1.1.4, we have $\lambda_+(f_0, g_0') = \lambda_-(f_0', f_0) = \beta$.

The following elementary properties of the quadratic numerical range with respect to certain transformations of the block operator matrix $\mathcal{A}$ are easy to check.

**Proposition 1.1.7** *We have*

i) $W^2(\alpha\mathcal{A} + \beta) = \alpha W^2(\mathcal{A}) + \beta$ *for* $\alpha,\ \beta \in \mathbb{C}$,
ii) $W^2(U^{-1}\mathcal{A}U) = W^2(\mathcal{A})$ *for* $U = \operatorname{diag}\big(U_1, U_2\big)$, $U_1 \in L(\mathcal{H}_1)$, $U_2 \in L(\mathcal{H}_2)$ *unitary.*

***Proof.*** Claim i) follows from the fact that $(\alpha\mathcal{A} + \beta)_{f,g} = \alpha\mathcal{A}_{f,g} + \beta$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. The assertion in ii) is a consequence of the equivalence $(f\ \ g)^{\mathrm{t}} \in S_{\mathcal{H}_1} \oplus S_{\mathcal{H}_2} \iff (U_1 f\ \ U_2 g)^{\mathrm{t}} \in S_{\mathcal{H}_1} \oplus S_{\mathcal{H}_2}$ and of the relation $(U^{-1}\mathcal{A}U)_{f,g} = \mathcal{A}_{U_1 f, U_2 g}$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. □

The first non-trivial property of the quadratic numerical range is that it is contained in the numerical range.

**Theorem 1.1.8** $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

***Proof.*** Let $\lambda_0 \in W^2(\mathcal{A})$. Then, by definition (1.1.5), there exist $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, and $(\alpha_1\ \alpha_2)^{\mathrm{t}} \in \mathbb{C}^2$, $|\alpha_1|^2 + |\alpha_2|^2 = 1$, such that

$$\mathcal{A}_{f,g}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \lambda_0\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

Taking the scalar product with $(\alpha_1\ \alpha_2)^{\mathrm{t}}$ and observing the definition of $\mathcal{A}_{f,g}$ in (1.1.4), we obtain

$$\left(\mathcal{A}\begin{pmatrix}\alpha_1 f\\ \alpha_2 g\end{pmatrix}, \begin{pmatrix}\alpha_1 f\\ \alpha_2 g\end{pmatrix}\right) = \lambda_0.$$

Since $\|\alpha_1 f\|^2 + \|\alpha_2 g\|^2 = 1$, this implies that $\lambda_0 \in W(\mathcal{A})$. □

Another feature of the quadratic numerical range is that the numerical ranges of the diagonal elements $W(A)$ and $W(D)$ are contained in $W^2(\mathcal{A})$ if the dimensions of $\mathcal{H}_1$ and $\mathcal{H}_2$ are at least two, more exactly:

**Theorem 1.1.9** *We have*

i) $\dim \mathcal{H}_2 \geq 2 \implies W(A) \subset W^2(\mathcal{A})$,
ii) $\dim \mathcal{H}_1 \geq 2 \implies W(D) \subset W^2(\mathcal{A})$.

***Proof.*** Let $f \in S_{\mathcal{H}_1}$ be arbitrary. If $\dim \mathcal{H}_2 \geq 2$, then there exists a $g \in S_{\mathcal{H}_2}$ such that $(Cf, g) = 0$. Thus

$$\mathcal{A}_{f,g} = \begin{pmatrix} (Af,f) & (Bg,f) \\ 0 & (Dg,g) \end{pmatrix}$$

and hence $(Af, f) \in \sigma_{\mathrm{p}}(\mathcal{A}_{f,g}) \subset W^2(\mathcal{A})$. The proof for $W(D)$ is similar. □

**Corollary 1.1.10** *Suppose that* $\dim \mathcal{H}_1 \geq 2$ *and* $\dim \mathcal{H}_2 \geq 2$.

i) *If* $W^2(\mathcal{A})$ *consists of two disjoint components,* $W^2(\mathcal{A}) = \mathcal{F}_1 \dot\cup \mathcal{F}_2$*, they can be enumerated such that*

$$W(A) \subset \mathcal{F}_1, \quad W(D) \subset \mathcal{F}_2.$$

ii) *If* $W(A) \cap W(D) \neq \emptyset$*, then* $W^2(\mathcal{A})$ *consists of only one component.*

***Proof.*** By the assumptions on the dimensions of $\mathcal{H}_1$ and $\mathcal{H}_2$, there exist $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ with $(Cf, g) = 0$. The eigenvalues of the corresponding matrix $\mathcal{A}_{f,g}$ are $(Af, f)$ and $(Dg, g)$; they belong to different components of $W^2(\mathcal{A})$ if the latter consists of two disjoint components. Theorem 1.1.9 and the fact that the numerical ranges $W(A)$ and $W(D)$ are connected (even convex) now imply claims i) and ii). □

The inclusions in Theorem 1.1.9 need not be true if $\dim \mathcal{H}_1 = 1$ or $\dim \mathcal{H}_2 = 1$:

**Example 1.1.11** Consider the $4 \times 4$ matrix from Example 1.1.2 with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}^1$,

$$\mathcal{A}_0 := \left(\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ -2 & -1 & 0 & -3\mathrm{i} \\ \hline -1 & -2 & 3\mathrm{i} & 0 \end{array}\right).$$

Figure 1.3 illustrates that the numerical range of the left upper corner of $\mathcal{A}_0$ is not contained in $W^2(\mathcal{A})$.

The property that $W^2(\mathcal{A})$ (or even its closure $\overline{W^2(\mathcal{A})}$) consists of two disjoint components will be of particular interest in the following sections. In this respect, the following results are useful.

**Proposition 1.1.12** *If* $\overline{W(A)} \cap \overline{W(D)} = \emptyset$ *and*

$$2\sqrt{\|B\|\,\|C\|} < \operatorname{dist}\big(W(A), W(D)\big),$$

*then* $\overline{W^2(\mathcal{A})}$ *consists of two disjoint components.*

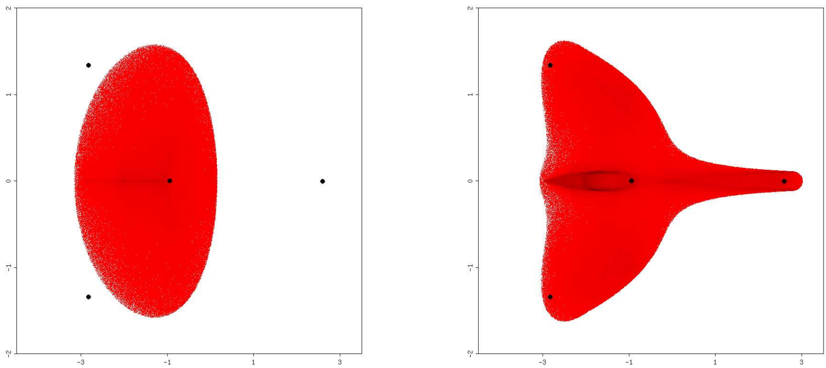

Figure 1.3 Numerical range of left upper corner and quadratic numerical range of $\mathcal{A}_0$.

***Proof.*** Set $\beta := \operatorname{dist}\big(W(A), W(D)\big)$ and assume that $\lambda$ belongs to the line that separates the convex sets $W(A)$ and $W(D)$ and has distance $\beta/2$ to both of them. Then, for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\begin{aligned}\big|\det\left(\mathcal{A}_{f,g} - \lambda\right)\big| &= \big|\big(\lambda - (Af,f)\big)\big(\lambda - (Dg,g)\big) - (Bg,f)(Cf,g)\big| \\ &\geq \big|\lambda - |(Af,f)|\big|\,\big|\lambda - |(Dg,g)|\big| - \|B\|\,\|C\| \\ &\geq \frac{\beta^2}{4} - \|B\|\,\|C\| > 0,\end{aligned}$$

which shows that $\lambda \notin W^2(\mathcal{A})$. □

The numerical range $W(\mathcal{A})$ of a bounded linear operator $\mathcal{A}$ is real if and only if $\mathcal{A}$ is self-adjoint. For the quadratic numerical range, we only have the following property.

**Proposition 1.1.13** *If $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, then*

i) $W^2(\mathcal{A}^*) = \big\{\overline{\lambda} \in \mathbb{C} : \lambda \in W^2(\mathcal{A})\big\} =: W^2(\mathcal{A})^*$,
ii) $\mathcal{A} = \mathcal{A}^* \implies W^2(\mathcal{A}) \subset \mathbb{R}$.

***Proof.*** Assertion i) follows from $(\mathcal{A}_{f,g})^* = (\mathcal{A}^*)_{f,g}$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$; claim ii) is obvious since in this case all matrices $\mathcal{A}_{f,g}$ are symmetric. □

If the quadratic numerical range is real, then, in the generic case, only self-adjointness with respect to a possibly indefinite inner product holds.

The corresponding notion of $\mathcal{J}$-self-adjointness plays a role in a number of other subsections and also in the next chapter on unbounded block operator matrices; therefore we give the definition for the unbounded case here.

**Definition 1.1.14** Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $\mathcal{J} \in L(\mathcal{H})$ have the corresponding block operator representation

$$\mathcal{J} := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.1.10}$$

A densely defined linear operator $\mathcal{A}$ in $\mathcal{H}$ is called *$\mathcal{J}$-self-adjoint* if $\mathcal{J}\mathcal{A}$ is self-adjoint in $\mathcal{H}$; it is called *$\mathcal{J}$-symmetric* if $\mathcal{J}\mathcal{A}$ is symmetric in $\mathcal{H}$.

Clearly, every bounded $\mathcal{J}$-symmetric operator is $\mathcal{J}$-self-adjoint. If we define the indefinite inner product $[\cdot,\cdot] := (\mathcal{J}\cdot,\cdot)$ on $\mathcal{H}$, then $\mathcal{A} \in L(\mathcal{H})$ is $\mathcal{J}$-self-adjoint if and only if

$$[\mathcal{A}\mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathcal{A}\mathbf{y}], \quad \mathbf{x}, \mathbf{y} \in \mathcal{H}.$$

The Hilbert space $\mathcal{H}$ equipped with the indefinite inner product $[\cdot,\cdot]$ is a Krein space; every $\mathcal{J}$-self-adjoint operator is a self-adjoint operator in this Krein space. For the definition of Krein spaces and properties of linear operators therein we refer to [Bog74], [AI89], [Lan82]. We only mention that the spectrum of a $\mathcal{J}$-self-adjoint operator is symmetric to $\mathbb{R}$.

Obviously, if $\mathcal{A} \in L(\mathcal{H})$ has a block operator representation (1.1.3), then

$$\begin{aligned} &\mathcal{A} \text{ is self-adjoint} && \iff && A = A^*, \quad D = D^*, \quad C = B^*, \\ &\mathcal{A} \text{ is } \mathcal{J}\text{-self-adjoint} && \iff && A = A^*, \quad D = D^*, \quad C = -B^*. \end{aligned}$$

**Theorem 1.1.15** *Let either* $\dim \mathcal{H}_1 \geq 2$ *or* $\dim \mathcal{H}_2 \geq 2$. *If* $W^2(\mathcal{A}) \subset \mathbb{R}$, *then* $A = A^*$, $D = D^*$, *and* $\mathcal{A}$ *is either block triangular* (*i.e.* $B = 0$ *or* $C = 0$) *or there exists a* $\gamma \in \mathbb{R}$, $\gamma \neq 0$, *such that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ \gamma B^* & D \end{pmatrix};$$

*in the latter case,* $\mathcal{A}$ *is similar to the block operator matrix*

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & \widetilde{B} \\ (\operatorname{sign}\gamma)\, \widetilde{B}^* & D \end{pmatrix}, \quad \widetilde{B} := \sqrt{|\gamma|}\, B;$$

$\widetilde{\mathcal{A}}$ *is self-adjoint in* $\mathcal{H}$ *if* $\operatorname{sign}\gamma = 1$ *and* $\mathcal{J}$*-self-adjoint if* $\operatorname{sign}\gamma = -1$.

In the proof of Theorem 1.1.15 we use the following lemma; in view of the next chapter, we formulate it for unbounded operators.

**Lemma 1.1.16** *If* $B$ *and* $C$ *are closed densely defined linear operators from* $\mathcal{H}_2$ *to* $\mathcal{H}_1$ *and from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$, *respectively, such that*

$$(By, x)(Cx, y) \in \mathbb{R} \quad \text{for all} \;\; x \in \mathcal{D}(C),\, y \in \mathcal{D}(B), \tag{1.1.11}$$

*then* $B = 0$, $C = 0$, *or* $C \subset \gamma B^*$ *with* $\gamma \in \mathbb{R}$ ($C = \gamma B^*$ *if* $B$, $C$ *are bounded*).

***Proof.*** If $x \in \mathcal{D}(C)$, $y \in \mathcal{D}(B)$ are such that $(By,x) \neq 0$, then condition (1.1.11) implies that

$$\frac{(Cx,y)}{(x,By)} = \frac{(By,x)(Cx,y)}{(By,x)(x,By)} \in \mathbb{R}. \tag{1.1.12}$$

Assume that $B \neq 0$. Then, because $C$ is densely defined, there exist elements $x_0 \in \mathcal{D}(C)$, $y_0 \in \mathcal{D}(B)$ such that $(x_0, By_0) \neq 0$. For $u \in \mathcal{D}(C)$, $v \in \mathcal{D}(B)$, we consider the function

$$\begin{aligned} f_{u,v}(z) :=& \frac{\big(C(x_0+zu), y_0+\overline{z}\,v\big)}{\big(x_0+zu, B(y_0+\overline{z}\,v)\big)} \\ =& \frac{(Cx_0,y_0) + z\big((Cx_0,v)+(Cu,y_0)\big) + z^2(Cu,v)}{(x_0,By_0) + z\big((x_0,Bv)+(u,By_0)\big) + z^2(u,Bv)}, \quad z \in \mathbb{C}. \end{aligned}$$

Since $(x_0, By_0) \neq 0$, the denominator is not identically zero and hence the function $f_{u,v}$ is rational in $\mathbb{C}$ with at most two poles, say $\zeta_1$, $\zeta_2$. Because of (1.1.12), it is real on its domain of holomorphy and thus constant there: $f_{u,v}(z) = f_{u,v}(0) = (Cx_0,y_0)/(x_0,By_0) =: \gamma \in \mathbb{R}$, or

$$\begin{aligned} &(Cx_0,y_0) + z\big((Cx_0,v)+(Cu,y_0)\big) + z^2(Cu,v) \\ &= \gamma\,\big((x_0,By_0) + z\big((x_0,Bv)+(u,By_0)\big) + z^2(u,Bv)\big) \end{aligned}$$

for $z \in \mathbb{C} \setminus \{\zeta_1,\zeta_2\}$. Comparing coefficients, we find $(Cu,v) = \gamma(u,Bv)$ for all $u \in \mathcal{D}(C)$, $v \in \mathcal{D}(B)$ and hence $\gamma B \subset C^*$ or, taking adjoints, $C \subset \gamma B^*$. The last claim is obvious. □

***Proof of Theorem* 1.1.15.** Without loss of generality, let $\dim \mathcal{H}_2 \geq 2$. Then, by Theorem 1.1.9, $W(A) \subset W^2(\mathcal{A}) \subset \mathbb{R}$ and hence $A$ is self-adjoint. This and the equality

$$(Af,f) + (Dg,g) = \lambda_+\begin{pmatrix} f \\ g \end{pmatrix} + \lambda_-\begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{R}, \quad f \in S_{\mathcal{H}_1},\ g \in S_{\mathcal{H}_2},$$

show that $D$ is self-adjoint as well. Since

$$\det \mathcal{A}_{f,g} = (Af,f)(Dg,g) - (Bg,f)(Cf,g) = \lambda_+\begin{pmatrix} f \\ g \end{pmatrix}\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{R}$$

for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, we have $(Bg,f)(Cf,g) \in \mathbb{R}$ for all $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$. Now Lemma 1.1.16 yields the second claim. The last assertion about the similarity of $\mathcal{A}$ follows from the identity

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & \sqrt{|\gamma|} \end{pmatrix} \begin{pmatrix} A & \widetilde{B} \\ (\operatorname{sign}\gamma)\,\widetilde{B}^* & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sqrt{|\gamma|}^{\,-1} \end{pmatrix}.$$

Obviously, $\widetilde{\mathcal{A}}$ is self-adjoint in $\mathcal{H}$ if $\operatorname{sign}\gamma = 1$; if $\operatorname{sign}\gamma = -1$, then $\mathcal{J}\widetilde{\mathcal{A}}$ is self-adjoint in $\mathcal{H}$ since

$$\mathcal{J}\widetilde{\mathcal{A}} = \begin{pmatrix} A & \widetilde{B} \\ \widetilde{B}^* & -D \end{pmatrix}. \qquad \square$$

## 1.2 Special classes of block operator matrices

A major advantage of the quadratic numerical range is that it reflects symmetries and other properties of the entries of a block operator matrix. Some of the results obtained here also play a role in the unbounded case considered in the next chapter. Therefore special emphasis is placed on structures occurring in applications *e.g.* from mathematical physics or systems theory.

**Theorem 1.2.1** *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}.$$

*For $\omega \in [0, \pi)$, define the sector $\Sigma_\omega := \{re^{i\phi} : r \geq 0, |\phi| \leq \omega\}$. If there exist $\alpha$, $\delta > 0$ and angles $\varphi, \vartheta \in [0, \pi/2]$ such that*

$$W(D) \subset \{z \in -\Sigma_\varphi : \operatorname{Re} z \leq -\delta\}, \quad W(A) \subset \{z \in \Sigma_\vartheta : \operatorname{Re} z \geq \alpha\}$$

*and $\theta := \max\{\varphi, \vartheta\}$, then*

$$W^2(\mathcal{A}) \subset \{z \in -\Sigma_\theta : \operatorname{Re} z \leq -\delta\} \cup \{z \in \Sigma_\theta : \operatorname{Re} z \geq \alpha\}$$

*consists of two components separated by the strip $\{z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha\}$.*

For the proof of this theorem we use the following elementary lemma for the eigenvalues of $2 \times 2$ matrices (see [LT98, Lemma 3.1]).

**Lemma 1.2.2** *Let $a, b, c, d \in \mathbb{C}$ be complex numbers with $\operatorname{Re} d < 0 < \operatorname{Re} a$ and $bc \geq 0$. Then the matrix*

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

*has eigenvalues $\lambda_1, \lambda_2$ such that:*

i) $\operatorname{Re} \lambda_2 \leq \operatorname{Re} d < 0 < \operatorname{Re} a \leq \operatorname{Re} \lambda_1$,
ii) $\min\{\operatorname{Im} a, \operatorname{Im} d\} \leq \operatorname{Im} \lambda_1, \quad \operatorname{Im} \lambda_2 \leq \max\{\operatorname{Im} a, \operatorname{Im} d\}$,
iii) $\lambda_1, -\lambda_2 \in \{z \in \mathbb{C} : |\arg z| \leq \max\{|\arg a|, \pi - |\arg d|\}\}$.

***Proof.*** We suppose that $\operatorname{Im} a \geq 0$ (otherwise we consider $\mathrm{A}^*$) and

$$\arg a \geq \pi - |\arg d| \tag{1.2.1}$$

(otherwise we start from $d$ instead of $a$ in the following). Assumption (1.2.1) implies that

$$\left|\frac{\operatorname{Im}(a-d)}{\operatorname{Re}(a-d)}\right| \leq \tan(\arg a). \tag{1.2.2}$$

The eigenvalues $\lambda_1, \lambda_2$ satisfy the equation

$$(a-\lambda)(d-\lambda) - t = 0, \quad t := bc \geq 0.$$

We consider them as functions $\lambda_{1,2}$ of $t$ and write

$$\lambda_{1,2}(t) - \frac{a+d}{2} = \pm\sqrt{\frac{(a-d)^2}{4} + t}, \quad t \geq 0. \tag{1.2.3}$$

Now we decompose $\lambda_i(t) =: x_i(t) + \mathrm{i}y_i(t)$, $i = 1, 2$, and $(a+d)/2 =: \beta + \mathrm{i}\gamma$ into real and imaginary parts. Squaring equation (1.2.3) and taking real and imaginary parts, we see that $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ satisfy the relations

$$\big(x(t)-\beta\big)^2 - \big(y(t)-\gamma\big)^2 = \frac{1}{4}\operatorname{Re}(a-d)^2 + t, \tag{1.2.4}$$

$$\big(x(t)-\beta\big)\big(y(t)-\gamma\big) = \frac{1}{8}\operatorname{Im}(a-d)^2. \tag{1.2.5}$$

The last equation shows that the eigenvalues $\lambda_1(t), \lambda_2(t)$ lie on a hyperbola with centre $\beta + \mathrm{i}\gamma = (a+d)/2$ and asymptotes $\operatorname{Im} z = \gamma$ and $\operatorname{Re} z = \beta$ parallel to the real and imaginary axis, the right hand branch passing through $a$ and the left hand branch through $d$. From the identity (1.2.4) it follows that for $0 \leq t \leq \infty$ the eigenvalues $\lambda_1(t)$ fill the part of the right hand branch which extends from $a$ to $\infty + \mathrm{i}\gamma$, and the eigenvalues $\lambda_2(t)$ fill the part of the left hand branch from $d$ to $-\infty + \mathrm{i}\gamma$. This implies i) and ii). In order to prove iii), it is sufficient to show that the derivatives of the hyperbola at $d$ and at $a$ are in modulus less than $\tan(\arg a)$. For example, for the derivative at $d$, it follows from (1.2.5) that

$$\frac{\dot{y}(0)}{\dot{x}(0)} = -\frac{y(0)-\gamma}{x(0)-\beta} = -\frac{\operatorname{Im} d - \frac{1}{2}\operatorname{Im}(a+d)}{\operatorname{Re} d - \frac{1}{2}\operatorname{Re}(a+d)} = -\frac{\operatorname{Im}(d-a)}{\operatorname{Re}(d-a)},$$

which is in modulus less than $\tan(\arg a)$ by (1.2.2). □

***Proof of Theorem* 1.2.1.** All assertions follow by applying Lemma 1.2.2 to the $2 \times 2$ matrices $\mathcal{A}_{f,g}$ defined in (1.1.4) for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. □

For self-adjoint block operator matrices, the following corollary is obvious from Theorem 1.2.1.

**Corollary 1.2.3** *Let $\mathcal{A} = \mathcal{A}^*$ and suppose that*

$$\sup W(D) < \inf W(A).$$

*Then $W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \,\dot\cup\, \Lambda_+(\mathcal{A})$ consists of two components satisfying*

$$\sup \Lambda_-(\mathcal{A}) \le \sup W(D) < \inf W(A) \le \inf \Lambda_+(\mathcal{A}).$$

In the following proposition we generalize this estimate to non-separated diagonal entries and we derive two-sided estimates for the outer end-points $\inf \Lambda_-(\mathcal{A})$ and $\sup \Lambda_+(\mathcal{A})$ of the quadratic numerical range (see [KMM07]).

**Proposition 1.2.4** *If $\mathcal{A} = \mathcal{A}^*$, then the quadratic numerical range $W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \cup \Lambda_+(\mathcal{A})$ satisfies the estimates*

$$\begin{aligned} \inf \Lambda_+(\mathcal{A}) &\ge \max\big\{\inf W(A),\, \inf W(D)\big\}, \\ \sup \Lambda_-(\mathcal{A}) &\le \min\big\{\sup W(A),\, \sup W(D)\big\}, \end{aligned}$$

*and*

$$\begin{aligned} \min\big\{\inf W(A), \inf W(D)\big\} - \delta_B^- &\le \inf \Lambda_-(\mathcal{A}) \le \min\big\{\inf W(A), \inf W(D)\big\}, \\ \max\big\{\sup W(A), \sup W(D)\big\} &\le \sup \Lambda_+(\mathcal{A}) \le \max\big\{\sup W(A), \sup W(D)\big\} + \delta_B^+, \end{aligned}$$

*where*

$$\begin{aligned} \delta_B^- &:= \|B\| \tan\left(\frac{1}{2} \arctan \frac{2\|B\|}{|\inf W(A) - \inf W(D)|}\right), \\ \delta_B^+ &:= \|B\| \tan\left(\frac{1}{2} \arctan \frac{2\|B\|}{|\sup W(A) - \sup W(D)|}\right); \end{aligned}$$

*if* $\inf W(A) = \inf W(D)$ *or* $\sup W(A) = \sup W(D)$, *we set* $\arctan \infty := \pi/2$.

***Proof.*** Since $\mathcal{A} = \mathcal{A}^*$, we have $C = B^*$. Then the definition of $\lambda_+$ in Corollary 1.1.4 shows that, for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\begin{aligned} \lambda_+\begin{pmatrix} f \\ g \end{pmatrix} &= \frac{(Af,f)+(Dg,g)}{2} + \sqrt{\left(\frac{(Af,f)-(Dg,g)}{2}\right)^2 + |(Bg,f)|^2} \qquad (1.2.6) \\ &\ge \frac{(Af,f)+(Dg,g)}{2} + \left|\frac{(Af,f)-(Dg,g)}{2}\right| \\ &= \max\big\{(Af,f), (Dg,g)\big\}. \end{aligned}$$

From this estimate we obtain

$$\begin{aligned} \inf \Lambda_+(\mathcal{A}) &\ge \max\big\{\inf W(A),\, \inf W(D)\big\}, \\ \sup \Lambda_+(\mathcal{A}) &\ge \max\big\{\sup W(A),\, \sup W(D)\big\}. \end{aligned}$$

The proof of the second inequality and of the right part of the third inequality is analogous.

For the proof of the remaining inequalities, we observe that the solutions (1.2.6) of the quadratic equations $\det(\mathcal{A}_{f,g} - \lambda) = 0$ defining the quadratic numerical range can also be written in the form

$$\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} = \min\bigl\{(Af,f),(Dg,g)\bigr\} - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{|(Af,f)-(Dg,g)|}\right),$$
$$\lambda_+\begin{pmatrix} f \\ g \end{pmatrix} = \max\bigl\{(Af,f),(Dg,g)\bigr\} + |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{|(Af,f)-(Dg,g)|}\right).$$

Without loss of generality, we assume that $\inf W(A) \geq \inf W(D)$; otherwise we reverse the components in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose that $(Af,f) \geq (Dg,g)$; then

$$\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} = (Dg,g) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Af,f)-(Dg,g)}\right).$$

We define the auxiliary function

$$h(t) := t - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Af,f)-t}\right), \quad t \in \mathbb{R}.$$

It is easy to see that $h$ is strictly monotonically increasing (with a jump of height $2|(Bg,f)|$ at the singularity $(Af,f)$); in fact, we have $h' \geq 1/2$. Hence

$$\begin{aligned}\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} &\geq \inf W(D) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Af,f)-\inf W(D)}\right) \\ &\geq \inf W(D) - \|B\| \tan\left(\frac{1}{2}\arctan\frac{2\|B\|}{\inf W(A)-\inf W(D)}\right)\end{aligned}$$

if $(Af,f) \geq (Dg,g)$. If $(Af,f) < (Dg,g)$, then we have the estimates $(Af,f) \geq \inf W(A) \geq \inf W(D)$ and $(Dg,g) > (Af,f) \geq \inf W(A)$. Thus, in the same way as above, we obtain

$$\begin{aligned}\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} &= (Af,f) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Dg,g)-(Af,f)}\right) \\ &\geq \inf W(D) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Dg,g)-\inf W(D)}\right) \\ &\geq \inf W(D) - \|B\| \tan\left(\frac{1}{2}\arctan\frac{2\|B\|}{\inf W(A)-\inf W(D)}\right).\end{aligned}$$

The estimate for $\lambda_+$ is proved analogously. □

In the following, for a self-adjoint operator $T$ in a Hilbert space $\mathcal{H}$ and a subinterval $I \subset \mathbb{R}$, we denote by $E_T(I)$ the spectral projection and by $\mathcal{L}_I(T) = E_T(I)\mathcal{H}$ the spectral subspace, respectively, corresponding to $I$.

**Remark 1.2.5** Suppose that $\mathcal{A} = \mathcal{A}^*$ and $\inf W(A) \neq \inf W(D)$ with

$$\begin{aligned}\dim \mathcal{L}_{(-\infty, \inf W(A)]}(D) \geq 2 \quad &\text{if} \quad \inf W(D) < \inf W(A),\\ \dim \mathcal{L}_{(-\infty, \inf W(D)]}(A) \geq 2 \quad &\text{if} \quad \inf W(A) < \inf W(D).\end{aligned}$$

Then

$$\inf \Lambda_+(\mathcal{A}) = \max\bigl\{\inf W(A),\, \inf W(D)\bigr\}.$$

In general, the strict inequality $\inf \Lambda_+(\mathcal{A}) > \max\bigl\{\inf W(A),\, \inf W(D)\bigr\}$ may occur. Analogous statements hold for $\sup \Lambda_-(\mathcal{A})$.

***Proof.*** Let $\inf W(D) < \inf W(A)$ and $\dim \mathcal{L}_{(-\infty, \inf W(A)]}(D) > 1$. Since $\inf W(A) \in \sigma(A)$, there exists a sequence $(x_n)_1^\infty \subset \mathcal{D}(A)$, $\|x_n\| = 1$, such that $\bigl\|\bigl(A - \inf W(A)\bigr)x_n\bigr\| \to 0$, $n \to \infty$. Due to the dimension condition, for each $n \in \mathbb{N}$ there exists $y_n \in \mathcal{L}_{(-\infty, \inf W(A)]}(D)$, $\|y_n\| = 1$, such that $(B^* x_n, y_n) = 0$. Then $(D y_n, y_n) \leq (A x_n, x_n)$ and hence, by (1.2.6),

$$\lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (A x_n, x_n) \longrightarrow \inf W(A), \quad n \to \infty.$$

This implies $\inf W(A) \in \overline{\Lambda_+(\mathcal{A})}$. Together with Proposition 1.2.4, the first assertion follows. An example for strict inequality is furnished by the matrix

$$\mathcal{A} := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{array}\right);$$

here $\max\bigl\{\inf W(A),\, \inf W(D)\bigr\} = \inf W(A) = -1$ and

$$\min \Lambda_+(\mathcal{A}) = \lambda_+\begin{pmatrix} e_2 \\ e_2 \end{pmatrix} = -\frac{3}{2} + \frac{1}{2}\sqrt{5} > -1 \quad \text{with} \quad e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad \square$$

Next we consider block operator matrices for which $C = -B^*$ and, more specifically, $\mathcal{J}$-self-adjoint block operator matrices (see Definition 1.1.14). We estimate their quadratic numerical range in the case when the off-diagonal element $B$ is sufficiently small. In the $\mathcal{J}$-self-adjoint case, the quadratic numerical range is real for small $B$, but it may become complex if $B$ is sufficiently large; more detailed estimates are given in Proposition 1.3.9.

**Proposition 1.2.6** *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$$

*and define*

$$a_- := \inf \operatorname{Re} W(A), \quad a_+ := \sup \operatorname{Re} W(A),$$
$$d_- := \inf \operatorname{Re} W(D), \quad d_+ := \sup \operatorname{Re} W(D).$$

*Then the quadratic numerical range of $\mathcal{A}$ satisfies the following estimates:*

i) $\min\{a_-, d_-\} \le \operatorname{Re} W^2(\mathcal{A}) \le \max\{a_+, d_+\}$.

ii) *If $d_+ < a_-$ and $\|B\| < (a_- - d_+)/2$, then $W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \dot\cup \Lambda_+(\mathcal{A})$ consists of two components satisfying*

$$\operatorname{Re} \Lambda_-(\mathcal{A}) \le d_+ + \|B\| < a_- - \|B\| \le \operatorname{Re} \Lambda_+(\mathcal{A}).$$

iii) *If $A = A^*$, $D = D^*$, then $W^2(\mathcal{A})$ is symmetric to $\mathbb{R}$, $|\operatorname{Im} W^2(\mathcal{A})| \le \|B\|$; if, in addition, $d_+ < a_-$, then*

$$\|B\| \le (a_- - d_+)/2 \implies W^2(\mathcal{A}) \subset \mathbb{R},$$
$$\|B\| > (a_- - d_+)/2 \implies \big|\operatorname{Im} W^2(\mathcal{A})\big| \le \sqrt{\|B\|^2 - \frac{(a_- - d_+)^2}{4}}.$$

*The case $a_+ < d_-$ in* ii) *and* iii) *is analogous.*

For the proof of Proposition 1.2.6, we use the following simple lemma (see [Tre08, Lemma 5.1 ii)]).

**Lemma 1.2.7** *Let $a, b, c, d \in \mathbb{C}$ be complex numbers with $\operatorname{Re} d < \operatorname{Re} a$ and $bc \le 0$. Then the matrix*

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

*has eigenvalues $\lambda_1, \lambda_2$ such that*

i) $\operatorname{Re} d \le \operatorname{Re} \lambda_2 \le \operatorname{Re} \lambda_1 \le \operatorname{Re} a$,

ii) $\operatorname{Re} \lambda_2 \le \operatorname{Re} d + \sqrt{|bc|} < \operatorname{Re} a - \sqrt{|bc|} \le \operatorname{Re} \lambda_1$ *if* $\sqrt{|bc|} < (\operatorname{Re} a - \operatorname{Re} d)/2$, *and $\lambda_1$, $\lambda_2 \in \mathbb{R}$ if, in addition, $a$, $d \in \mathbb{R}$,*

iii) $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = (a + d)/2$, $|\operatorname{Im} \lambda_1| = |\operatorname{Im} \lambda_2| = \sqrt{|bc| - (a-d)^2/4}$ *if* $\sqrt{|bc|} \ge (a-d)/2$ *and $a$, $d \in \mathbb{R}$.*

***Proof.*** i) If $\operatorname{Re} \lambda < \operatorname{Re} d$ ($< \operatorname{Re} a$) or $\operatorname{Re} \lambda > \operatorname{Re} a$ ($> \operatorname{Re} d$), then the eigenvalue equation $(a-\lambda)(d-\lambda) = bc \le 0$ cannot hold. In fact, decomposing all numbers therein into real and imaginary parts, one can show that $\operatorname{Im} a - \operatorname{Im} \lambda$ and $\operatorname{Im} d - \operatorname{Im} \lambda$ have different signs and $\operatorname{Re}\big((a-\lambda)(d-\lambda)\big) > 0$.

ii) If $\operatorname{Re} d + \sqrt{|bc|} < \operatorname{Re}\lambda < \operatorname{Re} a - \sqrt{|bc|}$, then

$$\big|\det(\mathrm{A}-\lambda)\big| \geq |a-\lambda|\,|d-\lambda| - |bc| \geq |\operatorname{Re} a - \operatorname{Re}\lambda|\,|\operatorname{Re} d - \operatorname{Re}\lambda| - |bc| > 0,$$

hence $\lambda$ is not an eigenvalue of A. The relation $\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 = \operatorname{Re} a + \operatorname{Re} d$ excludes the possibility that *e.g.* $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 \leq \operatorname{Re} d + \sqrt{|bc|}$.

The claims in ii) for $a$, $d \in \mathbb{R}$ and in iii) are immediate from the formula

$$\lambda_{1/2} = \frac{a+d}{2} \pm \sqrt{\frac{(a-d)^2}{4} + bc}\,. \qquad \square$$

***Proof of Proposition* 1.2.6.** If $A = A^*$, $D = D^*$, then $\overline{\det\big(\mathcal{A}_{f,g} - \lambda\big)} = \det\big(\mathcal{A}_{f,g} - \overline{\lambda}\big)$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, which implies $W^2(\mathcal{A}) = W^2(\mathcal{A})^*$ and hence the first claim in i). All other claims follow by applying Lemma 1.2.7 to the $2 \times 2$ matrices $\mathcal{A}_{f,g}$ defined in (1.1.4) for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. $\square$

**Proposition 1.2.8** *Let $\mathcal{H}_1 = \mathcal{H}_2$ and suppose that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & A^* \end{pmatrix}$$

*is such that either $B = B^*$ and $C = C^*$ or $B = -B^*$ and $C = -C^*$. Then $W^2(\mathcal{A})$ is symmetric to $\mathbb{R}$.*

***Proof.*** For $f, g \in S_{\mathcal{H}_1}$ and $\lambda \in \mathbb{C}$, it is easy to see that

$$\begin{aligned}\det\big((\mathcal{A}^*)_{g,f} - \lambda\big) &= \det\begin{pmatrix} (A^*g,g)-\lambda & (C^*f,g) \\ (B^*g,f) & (Af,f)-\lambda \end{pmatrix} \\ &= \det\begin{pmatrix} (Af,f)-\lambda & \pm(Bg,f) \\ \pm(Cf,g) & (A^*g,g)-\lambda \end{pmatrix} = \det\big(\mathcal{A}_{f,g} - \lambda\big),\end{aligned}$$

which implies that $W^2(\mathcal{A}) = W^2(\mathcal{A}^*)$. On the other hand, by Proposition 1.1.13 i), we have $W^2(\mathcal{A}^*) = W^2(\mathcal{A})^*$ and hence $W^2(\mathcal{A}) = W^2(\mathcal{A})^*$. $\square$

**Proposition 1.2.9** *Let $\mathcal{H}_1 = \mathcal{H}_2$ and suppose that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$$

*is such that either $B = B^*$ and $C = C^*$ or $B = -B^*$ and $C = -C^*$. Then $W^2(\mathcal{A})$ is symmetric to $\mathrm{i}\mathbb{R}$.*

***Proof.*** The assertion follows since $\mathrm{i}\mathcal{A}$ satisfies the assumptions of Proposition 1.2.8 and $W^2(\mathrm{i}\mathcal{A}) = \mathrm{i}W^2(\mathcal{A})$ by Proposition 1.1.7 i). $\square$

## 1.3 Spectral inclusion

The most important feature of the quadratic numerical range is that, like the numerical range, it has the spectral inclusion property (see (1.1.1)). Since the quadratic numerical range is always contained in the numerical range (see Theorem 1.1.8), it furnishes a possibly tighter spectral enclosure.

**Theorem 1.3.1** $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}),\ \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$

For the proof of Theorem 1.3.1 we need a simple lemma about $2 \times 2$-matrices, which we prove for the convenience of the reader.

**Lemma 1.3.2** *If for $\mathcal{M} \in M_2(\mathbb{C})$ there exists a vector $x \in \mathbb{C}^2$ such that*

$$\|x\| = 1 \quad \textit{and} \quad \|\mathcal{M}x\| < \varepsilon, \tag{1.3.1}$$

*then* $\operatorname{dist}\bigl(0, \sigma(\mathcal{M})\bigr) \le \sqrt{\|\mathcal{M}\|\,\varepsilon}$.

***Proof.*** Only the case that the matrix $\mathcal{M}$ is invertible has to be considered. Then the inverse matrix $\mathcal{M}^{-1}$ can be written as

$$\mathcal{M}^{-1} = \frac{1}{\det \mathcal{M}} (J^{\mathrm{t}} \mathcal{M} J)^{\mathrm{t}}$$

with $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus

$$\|\mathcal{M}^{-1}\| = \frac{\|\mathcal{M}\|}{|\det \mathcal{M}|} = \frac{\|\mathcal{M}\|}{|\lambda_1 \lambda_2|}, \tag{1.3.2}$$

where $\lambda_1$, $\lambda_2$ are the eigenvalues of $\mathcal{M}$. The assumption (1.3.1) implies that

$$\|\mathcal{M}^{-1}\| > \varepsilon^{-1}. \tag{1.3.3}$$

From (1.3.2) and (1.3.3) we obtain $\min\bigl\{|\lambda_1|, |\lambda_2|\bigr\} \le \sqrt{\|\mathcal{M}\|\,\varepsilon}$. □

In the following, for a bounded or unbounded linear operator $T$ in $\mathcal{H}$, we define its *approximate point spectrum* $\sigma_{\mathrm{app}}(T)$ as

$$\sigma_{\mathrm{app}}(T) := \bigl\{\lambda \in \mathbb{C} : \exists\, (\mathbf{x}_n)_1^\infty \subset \mathcal{D}(T),\ \|\mathbf{x}_n\| = 1,\ (T-\lambda)\mathbf{x}_n \to 0,\ n \to \infty\bigr\}. \tag{1.3.4}$$

***Proof of Theorem* 1.3.1.** First we consider $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A})$. Then there exists a nonzero element $(f\ \ g)^{\mathrm{t}} \in \mathcal{H}$ such that

$$(A - \lambda) f + B g = 0,$$
$$C f + (D - \lambda) g = 0.$$

We write $f = \|f\|\,\widehat{f}$, $g = \|g\|\,\widehat{g}$ with elements $\widehat{f} \in S_{\mathcal{H}_1}$, $\widehat{g} \in S_{\mathcal{H}_2}$ (here, if *e.g.* $f = 0$, then $\widehat{f}$ can be chosen arbitrarily). It follows that

$$\bigl(Af, \widehat{f}\,\bigr) - \lambda\bigl(f, \widehat{f}\,\bigr) + \bigl(Bg, \widehat{f}\,\bigr) = 0,$$
$$\bigl(Cf, \widehat{g}\bigr) + \bigl(Dg, \widehat{g}\bigr) - \lambda\bigl(g, \widehat{g}\bigr) = 0,$$

and, consequently,

$$\mathcal{A}_{\widehat{f},\widehat{g}}\begin{pmatrix}\|f\|\\ \|g\|\end{pmatrix} = \lambda\begin{pmatrix}\|f\|\\ \|g\|\end{pmatrix}. \tag{1.3.5}$$

Hence $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{f},\widehat{g}}) \subset W^2(\mathcal{A})$.

If $\lambda \in \sigma(\mathcal{A}) \setminus \sigma_{\mathrm{p}}(\mathcal{A})$, then $\overline{\lambda} \in \sigma_{\mathrm{p}}(\mathcal{A}^*)$ or $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A})$. If $\overline{\lambda} \in \sigma_{\mathrm{p}}(\mathcal{A}^*)$, then, according to what was shown above, we have $\overline{\lambda} \in W^2(\mathcal{A}^*)$ and hence $\lambda \in W^2(\mathcal{A})$ by Proposition 1.1.13 i). If $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A})$, then there exists a sequence of elements $(f_n\ g_n)^{\mathrm{t}} \in \mathcal{H}$, $n = 1, 2, \dots$, such that

$$\|f_n\|^2 + \|g_n\|^2 = 1, \qquad \left\|\mathcal{A}\begin{pmatrix}f_n\\ g_n\end{pmatrix} - \lambda\begin{pmatrix}f_n\\ g_n\end{pmatrix}\right\| \longrightarrow 0, \quad n \to \infty.$$

Then, with $\widehat{f}_n \in S_{\mathcal{H}_1}$, $\widehat{g}_n \in S_{\mathcal{H}_2}$ as in the first part of the proof, we obtain

$$\mathcal{A}_{\widehat{f}_n,\widehat{g}_n}\begin{pmatrix}\|f_n\|\\ \|g_n\|\end{pmatrix} - \lambda\begin{pmatrix}\|f_n\|\\ \|g_n\|\end{pmatrix} \longrightarrow 0, \quad n \to \infty.$$

Since $\|\mathcal{A}_{\widehat{f}_n,\widehat{g}_n}\| \le \|\mathcal{A}\|$, $n \in \mathbb{N}$, we have $\operatorname{dist}\big(\lambda, \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{f}_n,\widehat{g}_n})\big) \to 0$ for $n \to \infty$ by Lemma 1.3.2. Thus $\lambda \in \overline{\bigcup_{n\in\mathbb{N}} \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{f}_n,\widehat{g}_n})} \subset \overline{W^2(\mathcal{A})}$. □

Altogether, in Theorem 1.3.1 and Theorem 1.1.8, we have shown that

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}) \subset W(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})}.$$

Therefore, and because of its non-convexity, the quadratic numerical range $W^2(\mathcal{A})$ may give better information about the localization of the spectrum $\sigma(\mathcal{A})$ than the numerical range $W(\mathcal{A})$.

**Example 1.3.3** Consider the $4 \times 4$ matrix

$$\mathcal{A}_3 := \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \hline -2 & -1 & \mathrm{i} & 5\mathrm{i}\\ -1 & -2 & -5\mathrm{i} & \mathrm{i}\end{array}\right)$$

with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. Figure 1.4 shows its numerical range, quadratic numerical range, and the four different eigenvalues marked by black dots, two in each component of $W^2(\mathcal{A}_3)$.

If $\mathcal{A}$ is self-adjoint, Theorems 1.3.1 and 1.1.9 allow to characterize the quadratic numerical range more explicitly.

**Proposition 1.3.4** *Suppose that* $\mathcal{A} = \mathcal{A}^*$ *and* $\dim \mathcal{H}_1 \ge 2$, $\dim \mathcal{H}_2 \ge 2$.

i) *If* $\overline{W(A)} \cap \overline{W(D)} \ne \emptyset$, *then* $\overline{W^2(\mathcal{A})}$ *is the single interval*

$$\overline{W^2(\mathcal{A})} = \big[\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A})\big] = \overline{W(\mathcal{A})}. \tag{1.3.6}$$

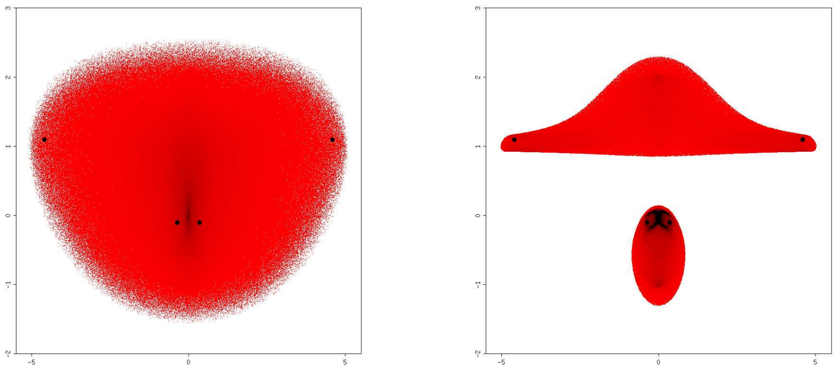


Figure 1.4 Numerical range, quadratic numerical range, and eigenvalues of $\mathcal{A}_3$.

ii) *If* $\overline{W(A)} \cap \overline{W(D)} = \emptyset$*, then* $\overline{W^2(\mathcal{A})}$ *consists of two disjoint intervals*

$$\overline{W^2(\mathcal{A})} = \big[\min \sigma(\mathcal{A}), d\big] \,\dot\cup\, \big[a, \max \sigma(\mathcal{A})\big] \tag{1.3.7}$$

*where*

$$d := \min\big\{\sup W(A),\ \sup W(D)\big\}, \tag{1.3.8}$$

$$a := \max\big\{\inf W(A),\ \inf W(D)\big\}. \tag{1.3.9}$$

***Proof.*** Theorem 1.1.8 implies that

$$\overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})} = \big[\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A})\big].$$

Since $\mathcal{A}$ is self-adjoint and $W^2(\mathcal{A})$ satisfies the spectral inclusion property by Theorem 1.3.1, we have

$$\min \sigma(\mathcal{A}),\ \max \sigma(\mathcal{A}) \in \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$$

Because $\overline{W^2(\mathcal{A})}$ consists of at most two connected sets, it follows that it is either of the form (1.3.6) or of the form (1.3.7).

Without loss of generality, we may assume that $\inf W(D) \le \inf W(A)$; otherwise we reverse the enumeration of the components in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then $\overline{W(A)} \cap \overline{W(D)} = \emptyset$ means that $\sup W(D) < \inf W(A)$. By Proposition 1.2.4, we conclude that

$$\sup \Lambda_-(\mathcal{A}) \le \sup W(D) < \inf W(A) \le \inf \Lambda_+(\mathcal{A}).$$

Since $\dim \mathcal{H}_1 \ge 2$, Theorem 1.1.9 applies and shows that $\sup W(D) \in \overline{W(D)} \subset \overline{W^2(\mathcal{A})} = \overline{\Lambda_-(\mathcal{A})} \,\dot\cup\, \overline{\Lambda_+(\mathcal{A})}$. Hence $d = \sup \Lambda_-(\mathcal{A}) = \sup W(D)$.

In the same way, it follows that $a = \inf \Lambda_+(\mathcal{A}) = \inf W(A)$ if $\dim \mathcal{H}_2 \geq 2$. This proves ii).

In order to show i), suppose to the contrary that $\overline{W^2(\mathcal{A})}$ consists of two disjoint intervals. Then, by Corollary 1.1.10 i), one of them contains $\overline{W(A)}$ and the other one $\overline{W(D)}$ so that $\overline{W(A)} \cap \overline{W(D)} = \emptyset$, a contradiction. □

**Remark 1.3.5** If $\mathcal{A} = \mathcal{A}^*$ and $\dim \mathcal{H}_1 = 1$, we only obtain "$\leq$" in (1.3.8); analogously, if $\dim \mathcal{H}_2 = 1$, we only obtain "$\geq$" in (1.3.9).

Note that if $\dim \mathcal{H}_1 = 1$ and $\dim \mathcal{H}_2 = 1$, then $\overline{W^2(\mathcal{A})} = W^2(\mathcal{A}) = \sigma_{\mathrm{p}}(\mathcal{A})$ consists of the eigenvalues of $\mathcal{A}$.

In the sequel, we present two theorems on the classical problem of perturbation of spectra of bounded self-adjoint operators (see [Dav63], [Dav65], [DK70]). The key tool is the spectral inclusion theorem for the quadratic numerical range, combined with the estimates given for it in Section 1.2.

First we consider arbitrary bounded self-adjoint operators subject to perturbations that are off-diagonal with respect to a certain decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the underlying Hilbert space (see [KMM07, Lemma 1.1], [LT98, Theorem 3.2]); the case that the spectrum of the unperturbed operator splits into two parts separated by a point is crucial in the following.

**Theorem 1.3.6** *If $\mathcal{A} = \mathcal{A}^*$, then $\sigma(\mathcal{A})$ satisfies the following estimates.*

i) *Define $\delta_B^{\pm}$ as in Proposition* 1.2.4*. Then*

$$\min\bigl\{\min \sigma(A), \min \sigma(D)\bigr\} - \delta_B^- \leq \min \sigma(\mathcal{A}) \leq \min\bigl\{\min \sigma(A), \min \sigma(D)\bigr\},$$
$$\max\bigl\{\max \sigma(A), \max \sigma(D)\bigr\} \leq \max \sigma(\mathcal{A}) \leq \max\bigl\{\max \sigma(A), \max \sigma(D)\bigr\} + \delta_B^+.$$

ii) *If* $\max \sigma(D) < \min \sigma(A)$, *then*

$$\sigma(\mathcal{A}) \cap \bigl(\max \sigma(D), \min \sigma(A)\bigr) = \emptyset$$

*independently of the norm of $B$.*


Figure 1.5 Enclosures for the spectra of $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and of $\left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)$.

***Proof.*** By Theorem 1.3.1 it is sufficient to prove that $W^2(\mathcal{A})$ satisfies the estimates claimed in i) and ii). This was proved in Proposition 1.2.4

for i) (observe that $\inf W(A) = \min \sigma(A)$, $\sup W(A) = \max \sigma(A)$, and analogously for $D$) and in Corollary 1.2.3 for ii). □

Next we consider the case that the spectra of the diagonal entries $A$ and $D$ are disjoint, *i.e.* their distance $\delta_{A,D}$ is positive. Classical perturbation theory yields that the spectrum of the perturbed operator $\mathcal{A}$ remains separated into two disjoint parts as long as $\|B\| < \delta_{A,D}/2$ (see [Kat95, Theorem V.4.10]). By means of Theorem 1.3.6, we are able to improve this result and derive an optimal bound on $\|B\|$ (see [KMM07, Theorem 1.3]).

**Theorem 1.3.7** *Let $\mathcal{A} = \mathcal{A}^*$, $\delta_{A,D} := \operatorname{dist}\big(\sigma(A), \sigma(D)\big) > 0$, and set*

$$\delta_B := \|B\| \tan\left(\frac{1}{2}\arctan\frac{2\|B\|}{\delta_{A,D}}\right).$$

i) *Then*

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A) \,\dot\cup\, \sigma(D)\big) \le \delta_B\big\}.$$

ii) *If $\|B\| < \frac{\sqrt{3}}{2}\,\delta_{A,D}$, then $\delta_B < \frac{1}{2}\delta_{A,D}$ and $\sigma(\mathcal{A}) = \sigma_1 \,\dot\cup\, \sigma_2$, $\sigma_1, \sigma_2 \neq \emptyset$, with*

$$\sigma_1 \subset \big\{\lambda\in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A)\big)\le \delta_B\big\} \subset \big\{\lambda\in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A)\big)< \delta_{A,D}/2\big\},$$
$$\sigma_2 \subset \big\{\lambda\in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(D)\big)\le \delta_B\big\} \subset \big\{\lambda\in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(D)\big)< \delta_{A,D}/2\big\}.$$

iii) *If $\big(\operatorname{conv} \sigma(A)\big) \cap \sigma(D) = \emptyset$ and $\|B\| < \sqrt{2}\,\delta_{A,D}$, then $\delta_B < \delta_{A,D}$ and $\sigma(\mathcal{A}) = \sigma_1 \,\dot\cup\, \sigma_2$, $\sigma_1, \sigma_2 \neq \emptyset$, with*

$$\sigma_1 \subset \big\{\lambda \in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A)\big)\le \delta_B\big\} \subset \big\{\lambda \in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A)\big)< \delta_{A,D}\big\},$$
$$\sigma_2 \subset \big\{\lambda \in \mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A)\big)\ge \delta_{A,D}\big\}.$$

***Proof.*** i) Let $\lambda \in \mathbb{R}$ be such that $\operatorname{dist}\big(\lambda, \sigma(A) \,\dot\cup\, \sigma(D)\big) > \delta_B$ and set $I_- := (-\infty, \lambda)$, $I_+ := (\lambda, \infty)$. If we write

$$\mathcal{A} = \mathcal{T} + \mathcal{S}, \quad \mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix},$$

then $\lambda \notin \sigma(\mathcal{T})$ and hence $\mathcal{H} = \mathcal{L}_{I_-}(\mathcal{T}) \oplus \mathcal{L}_{I_+}(\mathcal{T})$. If we denote $P_\pm := E_{\mathcal{T}}(I_\pm)$, then $\mathcal{L}_{I_\pm}(\mathcal{T}) = P_\pm\mathcal{H}$ and $P_-\mathcal{T}P_+ = 0$. Hence, with respect to the decomposition $\mathcal{H} = \mathcal{L}_{I_-}(\mathcal{T}) \oplus \mathcal{L}_{I_+}(\mathcal{T})$, the operator $\mathcal{A}$ can be written as

$$\mathcal{A} = \begin{pmatrix} P_-\mathcal{A}P_- & P_-\mathcal{A}P_+ \\ (P_-\mathcal{A}P_+)^* & P_+\mathcal{A}P_+ \end{pmatrix} = \begin{pmatrix} P_-\mathcal{A}P_- & P_-\mathcal{S}P_+ \\ (P_-\mathcal{S}P_+)^* & P_+\mathcal{A}P_+ \end{pmatrix}. \qquad (1.3.10)$$

If we further decompose $\mathcal{L}_{I_\pm}(\mathcal{T}) = \mathcal{L}_{I_\pm}(A) \oplus \mathcal{L}_{I_\pm}(D)$, then the diagonal elements in (1.3.10) have block operator representations

$$P_-\mathcal{A}P_- = \begin{pmatrix} A_- & B_- \\ B_-^* & D_- \end{pmatrix}, \quad P_+\mathcal{A}P_+ = \begin{pmatrix} A_+ & B_+ \\ B_+^* & D_+ \end{pmatrix}$$

with

$$A_\pm := E_A(I_\pm)AE_A(I_\pm),\ B_\pm := E_A(I_\pm)BE_D(I_\pm),\ D_\pm := E_D(I_\pm)DE_D(I_\pm).$$

Now Theorem 1.3.6 i), applied to the block operator matrices $P_-\mathcal{A}P_-$ and $P_+\mathcal{A}P_+$, shows that

$$\max\sigma(P_-\mathcal{A}P_-) \le \max\big\{\max\sigma(A_-), \max\sigma(D_-)\big\} + \delta_{B_-}^+, \tag{1.3.11}$$

$$\min\sigma(P_+\mathcal{A}P_+) \ge \min\big\{\min\sigma(A_+), \min\sigma(D_+)\big\} - \delta_{B_+}^- \tag{1.3.12}$$

where

$$\delta_{B_-}^+ := \|B_-\| \tan\left(\frac{1}{2}\arctan\frac{2\,\|B_-\|}{|\sup W(A_-) - \sup W(D_-)|}\right),$$

$$\delta_{B_+}^- := \|B_+\| \tan\left(\frac{1}{2}\arctan\frac{2\,\|B_+\|}{|\inf W(A_+) - \inf W(D_+)|}\right).$$

Obviously, $\sigma(A_\pm) \subset \sigma(A)$, $\sigma(D_\pm) \subset \sigma(D)$ so that

$$\begin{aligned}\big|\sup W(A_-) - \sup W(D_-)\big| &= \big|\max\sigma(A_-) - \max\sigma(D_-)\big| \\ &\ge \operatorname{dist}\big(\sigma(A_-), \max\sigma(D_-)\big) \\ &\ge \operatorname{dist}\big(\sigma(A), \max\sigma(D)\big) = \delta_{A,D};\end{aligned}$$

analogously, we see that $\big|\inf W(A_+) - \inf W(D_+)\big| \ge \delta_{A,D}$. Since the function

$$h(t) := t\tan\left(\frac{1}{2}\arctan(2t)\right), \quad t \in [0,\infty), \tag{1.3.13}$$

is strictly monotonically increasing and $\|B_\pm\| \le \|B\|$, we conclude that $\delta_{B_-}^+ \le \delta_B$ and $\delta_{B_+}^- \le \delta_B$. Furthermore, we have $\operatorname{dist}\big(\lambda, \sigma(A) \dot\cup \sigma(D)\big) > \delta_B$ by assumption and

$$\max\big\{\max\sigma(A_-), \max\sigma(D_-)\big\}, \min\big\{\min\sigma(A_+), \min\sigma(D_+)\big\} \in \sigma(A) \dot\cup \sigma(D).$$

This and the inequalities (1.3.11), (1.3.12) imply that

$$\max\sigma(P_-\mathcal{A}P_-) < \lambda < \min\sigma(P_+\mathcal{A}P_+).$$

Applying Theorem 1.3.6 ii) to $\mathcal{A}$ with respect to the block operator matrix representation (1.3.10), we conclude that

$$\lambda \in \big((\max\sigma(P_-\mathcal{A}P_-), \min\sigma(P_+\mathcal{A}P_+)\big) \subset \rho(\mathcal{A}).$$

For the proof of ii) and iii), we note that for the function $h$ defined in (1.3.13), we have $h(\sqrt{3}/2) = 1/2$, $h(\sqrt{2}) = 1$. Hence

$$\|B\| < \frac{\sqrt{3}}{2}\,\delta_{A,D} \implies \delta_B < \frac{\delta_{A,D}}{2}, \quad \|B\| < \sqrt{2}\,\delta_{A,D} \implies \delta_B < \delta_{A,D}. \tag{1.3.14}$$

Then ii) is immediate from i) and the first implication in (1.3.14). For the proof of iii), we observe that for $\sigma(\mathcal{A}) \cap \big(\operatorname{conv} \sigma(A)\big)$, the claim follows from i). For $\sigma(\mathcal{A}) \cap \big(\mathbb{R} \setminus \big(\operatorname{conv} \sigma(A)\big)\big)$, the claim follows if we show that

$$\big(\max \sigma(A) + \delta_B, \max \sigma(A) + \delta_{A,D}\big) \subset \rho(\mathcal{A}), \tag{1.3.15}$$

$$\big(\min \sigma(A) - \delta_{A,D}, \min \sigma(A) - \delta_B\big) \subset \rho(\mathcal{A}). \tag{1.3.16}$$

We prove (1.3.15); the proof of (1.3.16) is similar. We let $\lambda = \max \sigma(A) + \delta_B$ and proceed as in the proof of i). Then $E_A(I_+) = 0$ and thus

$$P_+ \mathcal{A} P_+ = \begin{pmatrix} 0 & 0 \\ 0 & D_+ \end{pmatrix}.$$

Using analogous estimates as in the proof of i) and the second implication in (1.3.14), we arrive at

$$\begin{aligned} \max \sigma(P_- \mathcal{A} P_-) < \lambda = \max \sigma(A) + \delta_B &< \max \sigma(A) + \delta_{A,D} \\ &\leq \min \sigma(D_+) = \min \sigma(P_+ \mathcal{A} P_+). \end{aligned}$$

Now Theorem 1.3.6 ii) shows that (1.3.15) holds. □

The following example illustrating Theorem 1.3.7 shows that the norm bounds therein are sharp.

**Example 1.3.8** Consider the family of $3 \times 3$ matrices

$$\mathcal{A}_\varepsilon := \left(\begin{array}{c|cc} 0 & \sqrt{2}\,\varepsilon & 0 \\ \hline \sqrt{2}\,\varepsilon & -1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad 0 \leq \varepsilon \leq 1,$$

with respect to the decomposition $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$. Then the spectra of the diagonal elements, $\{0\}$ and $\{-1, 1\}$, are disjoint and have distance 1. The eigenvalues of $\mathcal{A}_\varepsilon$ are given by

$$\lambda_1^\varepsilon = -\frac{1}{2} - \sqrt{2\,\varepsilon^2 + \frac{1}{4}}, \quad \lambda_2^\varepsilon = -\frac{1}{2} + \sqrt{2\,\varepsilon^2 + \frac{1}{4}}, \quad \lambda_3^\varepsilon = 1.$$

If $\varepsilon < \sqrt{3}/(2\sqrt{2})$, then the norm of the off-diagonal entry satisfies the assumption in Theorem 1.3.7 ii) which yields the inclusions

$$\sigma_1 = \{\lambda_2^\varepsilon\} \subset \big(-1/2, 1/2\big), \quad \sigma_2 = \{\lambda_1^\varepsilon, \lambda_3^\varepsilon\} \subset \big(-3/2, -1/2\big) \cup \big(1/2, 3/2\big);$$

if $\varepsilon = \sqrt{3}/(2\sqrt{2})$ and hence the norm of the off-diagonal entry reaches the critical value of the norm bound, then $\lambda_2^\varepsilon = 1/2$ and $\lambda_1^\varepsilon = -3/2$ reach the boundaries of the above inclusion intervals.

Since $\operatorname{conv}\{0\} \cap \{-1, 1\} = \emptyset$, Theorem 1.3.7 iii) applies as well. If $\varepsilon < 1$, then the norm of the off-diagonal entry satisfies the assumption in Theorem 1.3.7 iii) which yields the inclusions

$$\sigma_1 = \{\lambda_2^\varepsilon\} \subset \big(-1,1\big), \quad \sigma_2 = \{\lambda_1^\varepsilon, \lambda_3^\varepsilon\} \subset \big(-\infty,-1\big] \cup \big[1,\infty\big);$$

if $\varepsilon = 1$, then the norm of the off-diagonal entry reaches the critical value and we have $\lambda_2^\varepsilon = 1 = \lambda_3^\varepsilon$ and hence $\sigma_1$ and $\sigma_2$ are no longer disjoint.

If $\mathcal{A}$ is $\mathcal{J}$-self-adjoint, the spectral inclusion by the quadratic numerical range yields the following estimate for $\sigma(\mathcal{A})$ (see [LLMT05, Theorem 2.1] and [Tre08, Theorem 5.4]).

**Proposition 1.3.9** *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$$

*with $A = A^*$, $D = D^*$ and define*

$$a_- := \inf W(A), \quad a_+ := \sup W(A),$$
$$d_- := \inf W(D), \quad d_+ := \sup W(D).$$

*Then the spectrum of $\mathcal{A}$, which is symmetric to $\mathbb{R}$, satisfies the following estimates:*

i) $\sigma(\mathcal{A}) \cap \mathbb{R} \subset \overline{\operatorname{conv}\big(W(A) \cup W(D)\big)} = \big[\min\{a_-, d_-\}, \max\{a_+, d_+\}\big]$.

ii) $\sigma(\mathcal{A}) \setminus \mathbb{R} \subset \left\{ z \in \mathbb{C} \colon \dfrac{a_- + d_-}{2} \le \operatorname{Re} z \le \dfrac{a_+ + d_+}{2},\ |\operatorname{Im} z| \le \|B\| \right\}$.

iii) *If* $\delta := \operatorname{dist}\big(W(A), W(D)\big) = \min\big\{a_- - d_+, d_- - a_+\big\} > 0$, *then*

$$\|B\| \le \delta/2 \implies \sigma(\mathcal{A}) \subset \mathbb{R},$$
$$\|B\| > \delta/2 \implies \sigma(\mathcal{A}) \setminus \mathbb{R} \subset \left\{ z \in \mathbb{C} : |\operatorname{Im} z| \le \sqrt{\|B\|^2 - \delta^2/4} \right\}.$$

***Proof.*** Since the block operator matrix $\mathcal{A}$ is $\mathcal{J}$-self-adjoint (see Definition 1.1.14), the symmetry of $\sigma(\mathcal{A})$ to $\mathbb{R}$ is clear. All claims in i), ii), and iii) follow from the spectral inclusion in Theorem 1.3.1 and from the estimates for the quadratic numerical range in Proposition 1.2.6. □

**Example 1.3.10** The $4 \times 4$ matrix

$$\mathcal{A}_4 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ \hline 0 & -2 & -1 & 0 \\ -1 & -3 & 0 & 0 \end{array}\right)$$

satisfies the assumptions of Proposition 1.3.9 with $a_- = 0$, $a_+ = 1$ and $d_- = -1$, $d_+ = 0$; the inclusion in ii) therein yields that the non-real part of $W^2(\mathcal{A}_4)$ is confined to the strip $-1/2 \le \operatorname{Re} z \le 1/2$ (see Fig. 1.6) .

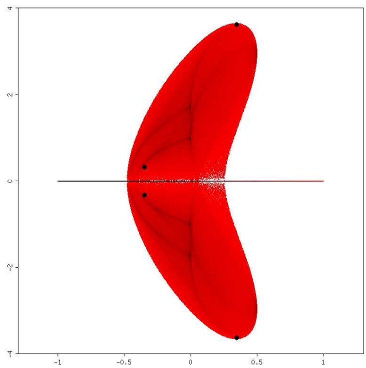


Figure 1.6 Quadratic numerical range of $\mathcal{A}_4$.

## 1.4 Estimates of the resolvent

The norm of the resolvent $(\mathcal{A}-\lambda)^{-1}$ of a bounded linear operator $\mathcal{A}$ can be estimated in terms of the numerical range as (see [Kat95, Theorem V.3.2])

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{1}{\operatorname{dist}\big(\lambda, W(\mathcal{A})\big)}, \quad \lambda \notin \overline{W(\mathcal{A})}.$$

The quadratic numerical range yields an analogous estimate in which the distance of $\lambda$ to $W^2(\mathcal{A})$ enters quadratically, not linearly.

**Theorem 1.4.1** *The resolvent of $\mathcal{A}$ admits the estimate*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\|\mathcal{A}\|+|\lambda|}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2}, \quad \lambda \notin \overline{W^2(\mathcal{A})}. \tag{1.4.1}$$

In the proof of this theorem we use the following lemma.

**Lemma 1.4.2** *If there exists a $\delta > 0$ such that for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$*

$$\|\mathcal{A}_{f,g}\,\alpha\| \ge \delta\,\|\alpha\|, \quad \alpha \in \mathbb{C}^2, \tag{1.4.2}$$

*then*

$$\|\mathcal{A}\mathbf{x}\| \ge \delta\,\|\mathbf{x}\|, \quad \mathbf{x} \in \mathcal{H}. \tag{1.4.3}$$

***Proof.*** Let $\mathbf{x} \in \mathcal{H}$. Then $\mathbf{x} = (\alpha_1 f \ \alpha_2 g)^{\mathrm{t}}$ with elements $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. For $\alpha = (\alpha_1 \ \alpha_2)^{\mathrm{t}} \in \mathbb{C}^2$ we have

$$\mathcal{A}_{f,g}\alpha = \begin{pmatrix} (Af,f)\alpha_1 + (Bg,f)\alpha_2 \\ (Cf,g)\alpha_1 + (Dg,g)\alpha_2 \end{pmatrix} = \begin{pmatrix} \big(A(\alpha_1 f) + B(\alpha_2 g), f\big) \\ \big(C(\alpha_1 f) + D(\alpha_2 g), g\big) \end{pmatrix}$$

and hence

$$\begin{aligned}\|\mathcal{A}_{f,g}\alpha\|^2 &= \big|\big(A(\alpha_1 f)+B(\alpha_2 g), f\big)\big|^2+\big|\big(C(\alpha_1 f)+D(\alpha_2 g), g\big)\big|^2 \\ &\le \|A(\alpha_1 f)+B(\alpha_2 g)\|^2+\|C(\alpha_1 f)+D(\alpha_2 g)\|^2=\|\mathcal{A}\mathbf{x}\|^2.\end{aligned} \tag{1.4.4}$$

Since $\|\mathbf{x}\|^2=|\alpha_1|^2+|\alpha_2|^2=\|\alpha\|^2$, (1.4.3) follows from (1.4.2) and (1.4.4). □

***Proof of Theorem* 1.4.1.** Let $\lambda \notin \overline{W^2(\mathcal{A})}$. Then the relation (1.3.2) implies that, for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\big\|(\mathcal{A}_{f,g}-\lambda)^{-1}\big\| = \frac{\|\mathcal{A}_{f,g}-\lambda\|}{|\det(\mathcal{A}_{f,g}-\lambda)|} = \frac{\|\mathcal{A}_{f,g}-\lambda\|}{\big|\lambda-\lambda_1\big(\begin{smallmatrix}f\\g\end{smallmatrix}\big)\big|\big|\lambda-\lambda_2\big(\begin{smallmatrix}f\\g\end{smallmatrix}\big)\big|}$$

where $\lambda_{1,2}\big(\begin{smallmatrix}f\\g\end{smallmatrix}\big) \in W^2(\mathcal{A})$ are the eigenvalues of $\mathcal{A}_{f,g}$. Since $\|\mathcal{A}_{f,g}\| \le \|\mathcal{A}\|$ and $\big|\lambda-\lambda_{1,2}\big(\begin{smallmatrix}f\\g\end{smallmatrix}\big)\big| \ge \operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)$, we find that, for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\big\|(\mathcal{A}_{f,g}-\lambda)^{-1}\big\| \le \frac{\|\mathcal{A}-\lambda\|}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2} \le \frac{\|\mathcal{A}\|+|\lambda|}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2}.$$

According to Lemma 1.4.2, this gives

$$\|(\mathcal{A}-\lambda)\mathbf{x}\| \ge \frac{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2}{\|\mathcal{A}\|+|\lambda|}\|\mathbf{x}\|, \quad \mathbf{x}\in\mathcal{H}.$$

Since $\lambda \notin \overline{W^2(\mathcal{A})}$ implies $\lambda \in \rho(\mathcal{A})$ by Theorem 1.3.1, (1.4.1) follows. □

The following example shows that, in general, the resolvent estimate in Theorem 1.4.1 cannot be improved.

**Example 1.4.3** Let $A=C=D=0$ and $B \in L(\mathcal{H}_2,\mathcal{H}_1)$, $B \not\equiv 0$. Then

$$\mathcal{A} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad W^2(\mathcal{A}) = \{0\},$$

and, for $\lambda \notin \overline{W^2(\mathcal{A})} = \{0\}$,

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| = \frac{1}{|\lambda|^2}\left\|\begin{pmatrix} -\lambda & -B \\ 0 & -\lambda \end{pmatrix}\right\| = \frac{1}{|\lambda|^2}\big(\|B\|+|\lambda|\big) = \frac{1}{|\lambda|^2}\big(\|\mathcal{A}\|+|\lambda|\big).$$

In a similar way as Theorem 1.4.1, the next two theorems can be proved.

**Theorem 1.4.4** *Suppose that there exists a subset $\mathcal{F} \subset W^2(\mathcal{A})$ such that for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ the matrix $\mathcal{A}_{f,g}$ has at most one eigenvalue in $\mathcal{F}$. Then, for all $\lambda \notin \overline{W^2(\mathcal{A})}$ such that* $\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\setminus\mathcal{F}\big) \ge \delta$ *with some $\delta > 0$, there exists a constant $\gamma(\delta) > 0$ (independent of $\lambda$) such that*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\gamma(\delta)}{\operatorname{dist}(\lambda,\mathcal{F})}.$$

**Theorem 1.4.5** *If* $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ *consists of two components, then*

$$\|(\mathcal{A}-\lambda)^{-1}\| \le \frac{\|\mathcal{A}\| + |\lambda|}{\operatorname{dist}(\lambda, \mathcal{F}_1)\operatorname{dist}(\lambda, \mathcal{F}_2)}, \qquad \lambda \notin \overline{W^2(\mathcal{A})}.$$

**Remark 1.4.6** The situation described in Theorem 1.4.4 occurs *e.g.* for $\mathcal{J}$-self-adjoint block operator matrices as in Proposition 1.3.9 provided that the numbers $a_\pm$ and $d_\pm$ defined therein satisfy $a_- < d_- < a_+ < d_+$. Then each matrix $\mathcal{A}_{f,g}$, $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ has at most one eigenvalue in each of the intervals $\big[a_-, (a_- + d_-)/2\big]$ and $\big[(a_+ + d_+)/2, d_+\big]$.

The resolvent estimate in terms of the numerical range implies that the length of a Jordan chain at an eigenvalue lying on the boundary of the numerical range is at most one, *i.e.* there are no associated vectors. As a corollary of Theorem 1.4.1, we obtain an analogue for boundary points of the quadratic numerical range. Since the latter is no longer convex, we need the following definition.

**Definition 1.4.7** Let $W \subset \mathbb{C}$. A boundary point $\lambda_0 \in \partial W$ is said to have the *exterior cone property* if there exists a closed cone $K$ (having positive aperture) with vertex $\lambda_0$ such that, for some $r > 0$,

$$K \cap \big\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \le r\big\} \cap \overline{W} = \{\lambda_0\}.$$

**Corollary 1.4.8** *Let* $\lambda_0 \in \sigma_{\rm p}(\mathcal{A})$. *If* $\lambda_0 \in \partial W^2(\mathcal{A})$ *has the exterior cone property, then the length of a Jordan chain at* $\lambda_0$ *is at most two.*

*If, in the situation of Theorem* 1.4.4, $\lambda_0 \in \partial\mathcal{F}$ *has the exterior cone property and* $\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\setminus\mathcal{F}\big) > 0$, *or, in the situation of Theorem* 1.4.5, $\lambda_0 \in \partial\mathcal{F}_1 \dot\cup \partial\mathcal{F}_2$ *has the exterior cone property, then the length of a Jordan chain at* $\lambda_0$ *is at most one, i.e. there are no associated vectors at* $\lambda_0$.

***Proof.*** Assume that there exists a Jordan chain $\{x_0, x_1, x_2\}$ of length 3 at $\lambda_0 \in \partial W^2(\mathcal{A})$. Then, for $\lambda \notin \overline{W^2(\mathcal{A})}$,

$$\|(\mathcal{A}-\lambda)^{-1}x_2\| = \left\|\frac{1}{(\lambda_0-\lambda)^3}x_0 + \frac{1}{(\lambda_0-\lambda)^2}x_1 + \frac{1}{\lambda_0-\lambda}x_2\right\| \ge C\frac{1}{|\lambda_0-\lambda|^3}$$

for $|\lambda_0 - \lambda|$ sufficiently small. If $\lambda_0$ has the exterior cone property and $\lambda$ lies on the axis of the cone $K$, $|\lambda_0 - \lambda| \le r$, then $|\lambda_0 - \lambda| \le C' \operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)$ with some constant $C' > 0$ and hence

$$\|(\mathcal{A}-\lambda)^{-1}x_2\| \ge C''\frac{1}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^3},$$

a contradiction. The proof of the other two assertions is similar. □

The following example shows that Jordan chains of length two may occur at boundary points of the quadratic numerical range.

**Example 1.4.9** In Example 1.4.3, the point 0 lies on the boundary of $W^2(\mathcal{A}) = \{0\}$ and has the exterior cone property. If we choose $g \in \mathcal{H}_2$ such that $Bg \neq 0$, then $(Bg\ 0)^{\mathrm{t}}$, $(0\ g)^{\mathrm{t}}$ is a Jordan chain of $\mathcal{A}$ at 0 of length two.

## 1.5 Corners of the quadratic numerical range

For a bounded linear operator $T$ in a Hilbert space it is well-known that a corner $\lambda_0 \in W(T)$ of the numerical range $W(T)$ is an eigenvalue of $T$. Moreover, every corner of $\lambda_0 \in \overline{W(T)}$ belongs to the spectrum of $T$ (see [Kat95], [HJ91], [GR97, Theorem 1.5-5, Corollary 1.5-6]).

For the quadratic numerical range these statements do not generalize in a straightforward way; this can be seen *e.g.* from the quadratic numerical range of the $4 \times 4$ matrix $\mathcal{A}_2$ in Example 1.1.5 which has 8 corners (see Fig. 1.2). It turns out that here not only the spectrum of the block operator matrix itself but also the spectra of its diagonal elements come into play.

We begin by giving the precise definition of a corner of a subset of $\mathbb{C}$.

**Definition 1.5.1** Let $W \subset \mathbb{C}$. A boundary point $\alpha \in \partial W$ is called *corner* of $W$ if there exist $\psi \in [0, \pi)$, $\varphi \in [0, 2\pi)$, and $\varepsilon > 0$ so that

$$\varphi \leq \arg(\lambda - \alpha) \leq \varphi + \psi, \quad \lambda \in W,\ |\lambda - \alpha| < \varepsilon, \tag{1.5.1}$$

where $\arg(\,\cdot\,)$ is suitably defined. The infimum $\psi_0$ of all $\psi \in [0, \pi)$ such that there exist $\varphi \in [0, 2\pi)$ and $\varepsilon > 0$ with (1.5.1) is called *angle* of the corner $\alpha$.

**Theorem 1.5.2** *Let $\lambda_0 \in W^2(\mathcal{A})$ and let $x_0 \in S_{\mathcal{H}_1}$, $y_0 \in S_{\mathcal{H}_2}$ be such that $\lambda_0$ is a zero of*

$$\Delta(x_0, y_0; \lambda) = \det \begin{pmatrix} (Ax_0, x_0) - \lambda & (By_0, x_0) \\ (Cx_0, y_0) & (Dy_0, y_0) - \lambda \end{pmatrix}. \tag{1.5.2}$$

*If $\lambda_0$ is a corner of $W^2(\mathcal{A})$, then at least one of the following holds:*

i) *$\lambda_0$ is an eigenvalue of $A$ with eigenvector $x_0$,*

ii) *$\lambda_0$ is an eigenvalue of $D$ with eigenvector $y_0$,*

iii) *$\lambda_0$ is an eigenvalue of $\mathcal{A}$ with eigenvector $(x_0\ \gamma y_0)^{\mathrm{t}}$ where*

$$\gamma = -\frac{(Cx_0, y_0)}{\big((D - \lambda_0)y_0, y_0\big)} \quad \textit{or} \quad \gamma = -\frac{\big((A - \lambda_0)x_0, x_0\big)}{(By_0, x_0)}.$$

***Proof.*** Without loss of generality, we assume that $\lambda_0 = 0$. First we consider the case of a simple zero. For $y \in S_{\mathcal{H}_2}$ and $z \in \mathbb{C}$ we define

$$\begin{aligned} g_y(\lambda, z) :=& \big((Ax_0, x_0) - \lambda\big)\Big(\big(D(y_0 + zy), y_0 + \overline{z}y\big) - \lambda(y_0 + zy, y_0 + \overline{z}y)\Big) \\ & - \big(B(y_0 + zy), x_0\big)\big(Cx_0, y_0 + \overline{z}y\big). \end{aligned}$$

Then $g_y(\lambda, 0) = \Delta(x_0, y_0; \lambda)$ and $g_y(\cdot, z)$ is a quadratic polynomial in $\lambda$. The latter has a zero $\lambda_y(z)$ such that $\lambda_y$ is analytic in a neighbourhood $U$ of 0 with $\lambda_y(0) = \lambda_0 = 0$ and which is given by

$$\begin{aligned} \lambda_y(z) = & \frac{(Ax_0, x_0)}{2} + \frac{\big(D(y_0+zy), y_0+\overline{z}y\big)}{2(y_0+zy, y_0+\overline{z}y)} \\ & + \sqrt{\left(\frac{(Ax_0, x_0)}{2} - \frac{\big(D(y_0+zy), y_0+\overline{z}y\big)}{2(y_0 + zy, y_0 + \overline{z}y)}\right)^2 + \frac{\big(B(y_0+zy), x_0\big)\big(Cx_0, y_0+\overline{z}y\big)}{4(y_0 + zy, y_0 + \overline{z}y)}}; \end{aligned} \tag{1.5.3}$$

here the branch of the square root is chosen such that $\lambda_y(0) = 0$. Obviously, $\lambda_y(t) \in \sigma_{\mathrm{p}}(\mathcal{A}_{x_0, y_0+ty}) \subset W^2(\mathcal{A})$ for real $t \in U$ and, by assumption, $\lambda_y(0) = 0$ is a corner of $W^2(\mathcal{A})$. This implies that the curve $\lambda_y(t)$, $t \in U \cap \mathbb{R}$, does not have a tangent in the point 0 and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_y(t)\bigg|_{t=0} = 0. \tag{1.5.4}$$

On the other hand, $g_y\big(\lambda_y(z), z\big) = 0$ for all $z \in \mathbb{C}$ and hence, for $t \in U \cap \mathbb{R}$,

$$\begin{aligned} 0 =& \frac{\mathrm{d}}{\mathrm{d}t}\, g_y\big(\lambda_y(t), t\big) \\ =& -\frac{\mathrm{d}}{\mathrm{d}t}\lambda_y(t)\Big(\big(D(y_0 + ty), y_0 + ty\big) - \lambda_y(t)\big(y_0 + ty, y_0 + ty\big)\Big) \\ & + \big((Ax_0, x_0) - \lambda_y(t)\big)\Big((Dy, y_0 + ty) + (D(y_0 + ty), y) \\ & - \frac{\mathrm{d}}{\mathrm{d}t}\lambda_y(t)\big(y_0 + ty, y_0 + ty\big) - \lambda_y(t)\big((y, y_0 + ty) + (y_0 + ty, y)\big)\Big) \\ & - \big(By, x_0\big)\big(Cx_0, y_0 + ty\big) - \big(B(y_0 + ty), x_0\big)\big(Cx_0, y\big). \end{aligned}$$

For $t = 0$ we obtain, together with (1.5.4) and $\lambda_y(0) = 0$,

$$\begin{aligned} 0 &= (Ax_0, x_0)\big((Dy, y_0) + (Dy_0, y)\big) - (By, x_0)(Cx_0, y_0) - (By_0, x_0)(Cx_0, y) \\ &= \big(y, \overline{(Ax_0, x_0)}D^*y_0 - \overline{(Cx_0, y_0)}B^*x_0\big) + \big((Ax_0, x_0)Dy_0 - (By_0, x_0)Cx_0, y\big). \end{aligned}$$

Since $y \in S_{\mathcal{H}_2}$ was arbitrary, the above relation also holds with $\mathrm{i}y$ instead of $y$ and so it follows that

$$(Ax_0, x_0)Dy_0 - (By_0, x_0)Cx_0 = 0, \tag{1.5.5}$$

$$\overline{(Ax_0, x_0)}D^*y_0 - \overline{(Cx_0, y_0)}B^*x_0 = 0. \tag{1.5.6}$$

In a similar way, for $x \in S_{\mathcal{H}_1}$ and $z \in \mathbb{C}$ we consider the polynomial

$$\begin{aligned} h_x(\lambda, z) :=& \big(\big(A(x_0 + zx), x_0 + \overline{z}x\big) - \lambda(x_0 + zx, x_0 + \overline{z}x)\big)\big((Dy_0, y_0) - \lambda\big) \\ &- \big(By_0, x_0 + \overline{z}x\big)\big(C(x_0 + zx), y_0\big) \end{aligned}$$

and arrive at

$$(Dy_0, y_0)Ax_0 - (Cx_0, y_0)By_0 = 0, \tag{1.5.7}$$

$$\overline{(Dy_0, y_0)}A^*x_0 - \overline{(By_0, x_0)}C^*y_0 = 0. \tag{1.5.8}$$

The numerical range $W(A)$ of $A$ is contained in $W^2(\mathcal{A})$ if $\dim \mathcal{H}_2 \geq 2$ (and analogously for $D$, see Theorem 1.1.9). We distinguish the following cases:

a) $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 1$: In this case $W^2(\mathcal{A})$ consists only of the two eigenvalues of $\mathcal{A}$, and the assertion is trivial.

b) $\dim \mathcal{H}_1 = 1$ or $\dim \mathcal{H}_2 = 1$: Let $\dim \mathcal{H}_2 = 1$; the case $\dim \mathcal{H}_1 = 1$ is analogous. Then $D$ is the multiplication by a constant, say $d$. If $Ax_0 = 0$ or $d = 0$, the corner 0 is an eigenvalue of $A$ or of $D$, respectively. If $Ax_0 \neq 0$ and $d \neq 0$, relation (1.5.7) yields that $(Cx_0, y_0) \neq 0$ and

$$Ax_0 + B\left(-\frac{(Cx_0, y_0)}{d}y_0\right) = 0.$$

Moreover, in fact $y_0 = 1$ and $Dy_0 = d$, so that we also have

$$Cx_0 + D\left(-\frac{(Cx_0, y_0)}{d}y_0\right) = 0.$$

Hence 0 is an eigenvalue of $\mathcal{A}$ with eigenvector

$$\begin{pmatrix} x_0 \\ -\dfrac{(Cx_0, y_0)}{(Dy_0, y_0)}y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ -\dfrac{1}{d}Cx_0 \end{pmatrix}.$$

Note that since $\dim \mathcal{H}_2 = 1$, we cannot conclude from $Ax_0 \neq 0$ that $(Ax_0, x_0) \neq 0$ and hence $(By_0, x_0) \neq 0$; the reason for this is that $(Ax_0, x_0) = 0$ shows that $0 \in W(A)$, but we cannot conclude that 0 is a corner of $W(A)$ because the numerical range of $A$ need not be contained in $W^2(\mathcal{A})$ (see Theorem 1.1.9). Therefore, in this case we can only use the first form of the constant $\gamma$ in the eigenvectors in iii).

c) $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$: First let $(Ax_0, x_0) = 0$. By Theorem 1.1.9, it follows that $W(A) \subset W^2(\mathcal{A})$ since $\dim\, \mathcal{H}_2 \geq 2$. Hence $0 \in W(A)$ is also a corner of $W(A)$. The well known theorem on corners of the numerical range (see [GR97, Theorem1.5-5]) now implies that $Ax_0 = 0$. If $(Dy_0, y_0) = 0$, a similar reasoning yields that $Dy_0 = 0$. If $(Ax_0, x_0) \neq 0$ and $(Dy_0, y_0) \neq 0$, then also $(By_0, x_0) \neq 0$ and (1.5.5), (1.5.7) imply that

$$Ax_0 + B\left(-\frac{(Cx_0, y_0)}{(Dy_0, y_0)}\, y_0\right) = 0,$$
$$Cx_0 + D\left(-\frac{(Ax_0, x_0)}{(By_0, x_0)}\, y_0\right) = 0.$$

Using the relation $(Ax_0, x_0)(Dy_0, y_0) - (By_0, x_0)(Cx_0, y_0) = 0$, we conclude that 0 is an eigenvalue of $\mathcal{A}$ with an eigenvector of the asserted form.

If $\lambda_0$ is a double zero, then, for every $y \in S_{\mathcal{H}_2}$, there are two root functions $\lambda_y^{(1)}(t)$, $\lambda_y^{(2)}(t)$, $t \in \mathbb{R}$, such that $g_y\big(\lambda_y^{(j)}(t), t\big) = 0$ near $t = 0$ and $\lambda_y^{(j)}(0) = 0$, $j = 1, 2$, with Puiseux expansions (see *e.g.* [Kat95, Section II.1.2])

$$\lambda_y^{(j)}(t) = \alpha_1 \mathrm{e}^{\pi \mathrm{i} j} t^{1/2} + \alpha_2 \mathrm{e}^{2\pi \mathrm{i} j} t + \dots, \quad j = 1, 2.$$

If $\alpha_1 \neq 0$, the four one-sided tangents of the functions $\lambda_y^{(1)}(t)$, $\lambda_y^{(2)}(t)$, $\lambda_y^{(1)}(-t)$, and $\lambda_y^{(2)}(-t)$, $t \geq 0$, divide the plane into four sectors of angle $\pi/2$. This contradicts the fact that 0 is a corner of $W^2(\mathcal{A})$. If $\alpha_1 = 0$, then $\lambda_y^{(j)}(t)$ are differentiable at 0 and the claim follows in the same way as in the case of a simple zero. □

**Remark 1.5.3** From equations (1.5.6) and (1.5.8), it follows that in case iii) of Theorem 1.5.2 the point $\overline{\lambda_0}$ is an eigenvalue of $\mathcal{A}^*$ with eigenvector $(x_0\ \widetilde{\gamma} y_0)^{\mathrm{t}}$ where

$$\widetilde{\gamma} = -\frac{(B^* x_0, y_0)}{\big((D^* - \overline{\lambda_0}) y_0, y_0\big)} \quad \text{or} \quad \widetilde{\gamma} = -\frac{\big((A^* - \overline{\lambda_0}) x_0, x_0\big)}{(C^* y_0, x_0)}.$$

**Remark 1.5.4** In order to prove relation (1.5.5), it is sufficient to consider roots $\lambda_{y_1}(t)$, $\lambda_{y_2}(t)$ (for real $t$) with

$$y_1 = (Ax_0, x_0) Dy_0 - (By_0, x_0) Cx_0 \quad \text{and} \quad y_2 = \mathrm{i} y_1,$$

and, for the proof of relation (1.5.7),

$$y_1 = (Dy_0, y_0) Ax_0 - (Cx_0, y_0) By_0 \;\text{ and }\; y_2 = \mathrm{i} y_1.$$

**Example 1.5.5** Consider the $4 \times 4$ matrices

$$\mathcal{A}_5 := \left(\begin{array}{cc|cc} 1 & 3+\mathrm{i} & 2 & \mathrm{i} \\ 3+\mathrm{i} & 1 & \mathrm{i} & 2 \\ \hline -2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \mathrm{i} & -2 & 3+\mathrm{i} & 1 \end{array}\right), \quad \mathcal{A}_6 := \left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right).$$

Their quadratic numerical ranges, displayed in Fig. 1.7, both have 6 corners: For $\mathcal{A}_5$ the four corners $-2-\mathrm{i}(1-\sqrt{5})$, $-2-\mathrm{i}(1+\sqrt{5})$, $4+\mathrm{i}(1+\sqrt{5})$, $4+\mathrm{i}(1-\sqrt{5})$ are the eigenvalues of $\mathcal{A}_5$ (marked by black dots), the corners $4+\mathrm{i}$, $-2-\mathrm{i}$ are the eigenvalues of the left upper corner $A$ (and, at the same time, of the right lower corner $D$). For $\mathcal{A}_6$ the four corners $-1+2\mathrm{i}$, $-1-2\mathrm{i}, 1+2\mathrm{i}, 1-2\mathrm{i}$ are the eigenvalues of $\mathcal{A}_6$ (marked by black dots), the corners $-1, 1$ are the eigenvalues of $A$ (and, at the same time, of $D$).

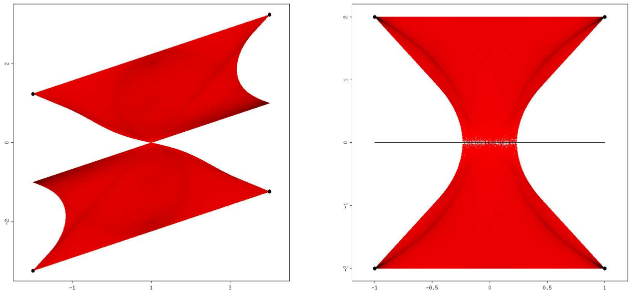

Figure 1.7 Quadratic numerical ranges of $\mathcal{A}_5$ and $\mathcal{A}_6$.

Next we consider corners of the quadratic numerical range $W^2(\mathcal{A})$ which do not belong to $W^2(\mathcal{A})$, but only to its closure. For this purpose, we use the well-known method of Banach limits (see [Ber62]). By passing to a Hilbert space formed by bounded sequences of $\mathcal{H}$, we convert points of the spectrum into eigenvalues of a corresponding linear operator and apply the previous Theorem 1.5.2 to the latter.

**Definition 1.5.6** Let $\mathcal{H}$ be an arbitrary Hilbert space, fix a Banach limit LIM on the space of bounded sequences in $\mathbb{C}$ with values in $\mathbb{C}$ (that is, a linear mapping which coincides with the usual limit for convergent sequences and is non-negative for non-negative sequences), let $\mathcal{R}$ be the linear space of all bounded sequences $x=(x_n)_1^\infty \subset \mathcal{H}$ with the (non-negative, but degenerate) inner product

$$[x,y] := \operatorname*{LIM}_{n\to\infty}\,(x_n,y_n), \quad x=(x_n)_1^\infty,\ y=(y_n)_1^\infty \in \mathcal{R},$$

and let $\mathcal{R}_0$ be the subspace of all $x=(x_n)_1^\infty \in \mathcal{R}$ with

$$\operatorname*{LIM}_{n\to\infty}\,(x_n,x_n)=0.$$

Then we define the Hilbert space $\widetilde{\mathcal{H}}$ as the completion of the quotient space $\mathcal{R}/\mathcal{R}_0$ with respect to the norm generated by the inner product $[\,\cdot\,,\,\cdot\,]$. For

Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, we associate with $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ the operator

$$\widetilde{T} \in L(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2), \quad \widetilde{T}\,\widetilde{x} := \big[(Tx_n)_1^\infty\big], \quad \widetilde{x} = (x_n)_1^\infty \in \widetilde{\mathcal{H}}_1,$$

where $[\,\cdot\,]$ denotes the equivalence class in $\mathcal{R}/\mathcal{R}_0$.

The following observations are easy to check (see [Ber62]).

**Remark 1.5.7** Let $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be Hilbert spaces.

i) The mapping $T \mapsto \widetilde{T}$ is an isometry from $L(\mathcal{H}_1, \mathcal{H}_2)$ into $L(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2)$.
ii) For $T \in L(\mathcal{H})$, we have $\sigma\big(\widetilde{T}\big) = \sigma_{\rm p}\big(\widetilde{T}\big) = \sigma_{\rm app}(T)$.

**Theorem 1.5.8** *If $\lambda_0 \in \overline{W^2(\mathcal{A})}$ is a corner of $W^2(\mathcal{A})$, then*

$$\lambda_0 \in \sigma(A) \cup \sigma(D) \cup \sigma(\mathcal{A}).$$

***Proof.*** Since $\lambda_0 \in \overline{W^2(\mathcal{A})}$, there exist a sequence $(\lambda_n)_1^\infty \subset W^2(\mathcal{A})$ with $\lambda_n \to \lambda_0$, $n \to \infty$, and sequences $(x_n^0)_1^\infty \subset S_{\mathcal{H}_1}$, $(y_n^0)_1^\infty \subset S_{\mathcal{H}_2}$ such that

$$\Delta(x_n^0, y_n^0; \lambda_n) = \det \begin{pmatrix} (Ax_n^0, x_n^0) - \lambda & (By_n^0, x_n^0) \\ (Cx_n^0, y_n^0) & (Dy_n^0, y_n^0) - \lambda \end{pmatrix} = 0.$$

We may assume that, for some neighbourhood $V$ of $\lambda_0$, all quadratic polynomials $\Delta(x_n^0, y_n^0; \,\cdot\,)$, $n \in \mathbb{N}$, have either one zero or two zeroes in $V$.

By means of a Banach limit, we construct the space $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$ and the operator

$$\widetilde{\mathcal{A}} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} : \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2 \to \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$$

according to Definition 1.5.6. First we consider the case of a simple zero. Since the sequences $(x_n^0)_1^\infty$ and $(y_n^0)_1^\infty$ are bounded, we may assume without loss of generality (by passing to suitable subsequences) that all sequences of the form

$$\big((Fu_n, v_n)\big)_1^\infty \tag{1.5.9}$$

converge where $F$ is one of the operators $A$, $B$, $C$, $D$, $A^*$, $B^*$, $C^*$, $D^*$ or a product of two or three of them, and $u_n$, $v_n$ are the elements $x_n^0$ or $y_n^0$, whenever the inner products in (1.5.9) are defined. Now let $\widetilde{x}^{\,0} = (x_n^0)_1^\infty \in \widetilde{\mathcal{H}}_1$, $\widetilde{y}^{\,0} = (y_n^0)_1^\infty \in \widetilde{\mathcal{H}}_2$. From Hurwitz's Theorem (see *e.g.* [Tit68, Chapter III, 3.45], it follows that $\lambda_0$ is a simple root of

$$\det \begin{pmatrix} (\widetilde{A}\widetilde{x}^{\,0}, \widetilde{x}^{\,0}) - \lambda & (\widetilde{B}\widetilde{y}^{\,0}, \widetilde{x}^{\,0}) \\ (\widetilde{C}\widetilde{x}^{\,0}, \widetilde{y}^{\,0}) & (\widetilde{D}\widetilde{y}^{\,0}, \widetilde{y}^{\,0}) - \lambda \end{pmatrix} = 0.$$

Hence $\lambda_0 \in W^2(\widetilde{\mathcal{A}})$.

Following the lines of the proof of Theorem 1.5.2, we derive the analogues of equalities (1.5.5), (1.5.7) for the operators $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$. We introduce the quadratic polynomial $g_{\widetilde{y}}(\lambda, z)$ with its root $\lambda_{\widetilde{y}}(z)$ given by a formula analogous to (1.5.3). By Remark 1.5.4, for the proof of the analogues of (1.5.5), (1.5.7), it is sufficient to consider *e.g.* elements $\widetilde{y} = (y_n)_1^\infty$ which are certain linear combinations of the four vectors $\widetilde{A}\widetilde{x}^0$, $\widetilde{B}\widetilde{y}^0$, $\widetilde{C}\widetilde{x}^0$, and $\widetilde{D}\widetilde{y}^0$. Since all sequences of the form (1.5.9) converge, we can use the ordinary limit instead of the Banach limit in the construction of the quadratic forms occurring in the formula for the root $\lambda_{\widetilde{y}}(z)$. This means that the root $\lambda_{\widetilde{y}}(z)$ is a limit of the corresponding roots $\lambda_{y_n}(z)$. By assumption, all roots $\lambda_y(t)$ for real $t \in U$ lie in a (closed) sector with vertex $\lambda_0$ and angle $< \pi$. Hence the roots $\lambda_{\widetilde{y}}(t)$ lie in the same sector, and we obtain the analogues of (1.5.5) and (1.5.7) for $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, $\widetilde{x}^0$, and $\widetilde{y}^0$. Now the proof for the case of a simple zero can be completed in a similar way as in the proof of Theorem 1.5.2; note that here we only have to consider case c) since $\dim \widetilde{\mathcal{H}}_1 = \dim \widetilde{\mathcal{H}}_2 = \infty$.

As a result of this and of Remark 1.5.7, we have $\lambda_0 \in \sigma_{\rm p}(\widetilde{A}) = \sigma_{\rm app}(A) \subset \sigma(A)$ or $\lambda_0 \in \sigma_{\rm p}(\widetilde{D}) = \sigma_{\rm app}(D) \subset \sigma(D)$ or $\lambda_0 \in \sigma_{\rm p}(\widetilde{\mathcal{A}}) = \sigma_{\rm app}(\mathcal{A}) \subset \sigma(\mathcal{A})$.

If $\lambda_0$ is a double zero, the proof is analogous to the corresponding part of the proof of Theorem 1.5.2. □

## 1.6 Schur complements and their factorization

In the spectral theory of block operator matrices an important role is played by the so-called Schur complements. For a $2 \times 2$ block operator matrix (1.1.3), there exist two Schur complements, one associated with each of the diagonal elements $A$ and $D$. The Schur complements are analytic operator functions defined outside of the spectrum of $D$ and of $A$, respectively.

First we prove that the numerical ranges of these analytic operator functions are contained in the quadratic numerical range. Further, we show that if the closure of the quadratic numerical range consists of two components, then a linear operator factor can be split off the Schur complements.

**Definition 1.6.1** For a block operator matrix $\mathcal{A}$ given by (1.1.3) the analytic operator functions $S_1 : \mathbb{C}\backslash\sigma(D) \to L(\mathcal{H}_1)$ and $S_2 : \mathbb{C}\backslash\sigma(A) \to L(\mathcal{H}_2)$,

$$S_1(\lambda) := A - \lambda - B(D-\lambda)^{-1}C, \quad \lambda \notin \sigma(D),$$
$$S_2(\lambda) := D - \lambda - C(A-\lambda)^{-1}B, \quad \lambda \notin \sigma(A),$$

are called *Schur complements* of $\mathcal{A}$.

If $\mathcal{H}$ is a Hilbert space, $\Omega \subset \mathbb{C}$ is open, and $S : \Omega \to L(\mathcal{H})$ is an analytic operator function, the resolvent set $\rho(S)$, the spectrum $\sigma(S)$, and the point spectrum $\sigma_{\mathrm{p}}(S)$ are defined as (see *e.g.* [Mar88, § 11.2])

$$\begin{aligned}\rho(S) &:= \{\lambda \in \Omega : S(\lambda) \text{ bijective in } \mathcal{H}\},\\ \sigma(S) &:= \Omega \setminus \rho(S),\\ \sigma_{\mathrm{p}}(S) &:= \{\lambda \in \Omega : S(\lambda) \text{ not injective in } \mathcal{H}\}.\end{aligned}$$

The numerical range $W(S)$ is defined as the set (see *e.g.* [Mar88, § 26.2])

$$W(S) = \big\{\lambda \in \Omega : \exists\, f \in \mathcal{H},\ f \neq 0,\ \big(S(\lambda)f, f\big) = 0\big\}. \tag{1.6.1}$$

Obviously, in the special case $S(\lambda) = T - \lambda$, $\lambda \in \mathbb{C}$, with a linear operator $T \in L(\mathcal{H})$, all these notions coincide with the usual definitions of the resolvent set, spectrum, point spectrum, and numerical range of the linear operator $T$.

It is well-known (see *e.g.* [Mar88, Theorem 26.6]) that $\sigma(S) \subset \overline{W(S)}$ if there exists a $\lambda_0 \in \Omega$ so that $0 \notin \overline{W(S(\lambda_0))}$. For the Schur complements, this condition is always satisfied for $\lambda_0$ large enough since

$$\|A\| + \|B(D-\lambda_0)^{-1}C\| < |\lambda_0| \quad \Longrightarrow \quad 0 \in \rho\big(S_1(\lambda_0)\big).$$

Hence $\sigma(S_1) \subset \overline{W(S_1)}$ and $\sigma(S_2) \subset \overline{W(S_2)}$.

The following *Frobenius-Schur factorization* of the block operator matrix $\mathcal{A}$ ties its spectral properties closely to those of its Schur complements.

**Proposition 1.6.2** *For $\lambda \notin \sigma(D)$ and $\lambda \notin \sigma(A)$, respectively, we have*

$$\mathcal{A} - \lambda = \begin{pmatrix} I & B(D-\lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D-\lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (D-\lambda)^{-1}C & I \end{pmatrix}, \tag{1.6.2}$$

$$\mathcal{A} - \lambda = \begin{pmatrix} I & 0 \\ C(A-\lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} A-\lambda & 0 \\ 0 & S_2(\lambda) \end{pmatrix} \begin{pmatrix} I & (A-\lambda)^{-1}B \\ 0 & I \end{pmatrix}, \tag{1.6.3}$$

*and hence*

$$\sigma(\mathcal{A}) \setminus \sigma(D) = \sigma(S_1), \quad \sigma(\mathcal{A}) \setminus \sigma(A) = \sigma(S_2).$$

**Theorem 1.6.3** *$W(S_1) \cup W(S_2) \subset W^2(\mathcal{A})$.*

***Proof.*** Let $\lambda \in W(S_1)$. Then there exists an $f \in \mathcal{H}_1$, $f \neq 0$, such that $\big(S_1(\lambda)f, f\big) = 0$. If $Cf = 0$, then $\big(S_1(\lambda)f, f\big) = \big((A-\lambda)f, f\big)$ and $\Delta(f, g; \lambda) = \big((A-\lambda)f, f\big)\big((D-\lambda)g, g\big)$. Thus $\Delta(f, g; \lambda) = 0$ for every

$g \in \mathcal{H}_2$, $g \neq 0$, and so $\lambda \in W^2(\mathcal{A})$. If $Cf \neq 0$, then $(D-\lambda)^{-1}Cf \neq 0$ and

$$\begin{aligned}\Delta\big(f,(D-\lambda)^{-1}Cf;\lambda\big) =& \big((A-\lambda)f,f\big)\big(Cf,(D-\lambda)^{-1}Cf\big) \\ &- \big(B(D-\lambda)^{-1}Cf,f\big)\big(Cf,(D-\lambda)^{-1}Cf\big) \\ =& \big(S_1(\lambda)f,f\big)\,\big(Cf,(D-\lambda)^{-1}Cf\big). \qquad (1.6.4)\end{aligned}$$

Hence $\big(S_1(\lambda)f,f\big) = 0$ implies that $\Delta\big(f,(D-\lambda)^{-1}Cf;\lambda\big) = 0$ and thus $\lambda \in W^2(\mathcal{A})$. The proof for $W(S_2)$ is similar. □

**Theorem 1.6.4** *Suppose that* $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$, *and assume that* $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$ *consists of two components. Then* $\mathcal{F}_1$, $\mathcal{F}_2$ *can be enumerated such that*

$$W(S_1) \cap \mathcal{F}_1 \neq \emptyset, \quad W(S_2) \cap \mathcal{F}_2 \neq \emptyset.$$

***Proof.*** Due to the dimension conditions, Corollary 1.1.10 i) shows that we can enumerate the components $\mathcal{F}_1$, $\mathcal{F}_2$ such that

$$\overline{W(A)} \subset \mathcal{F}_1, \quad \overline{W(D)} \subset \mathcal{F}_2. \qquad (1.6.5)$$

The claim is trivial if either $B = 0$ or $C = 0$; in this case $W(S_1) = W(A)$ and $W(S_2) = W(D)$. So we may assume that $B \neq 0$ and $C \neq 0$.

Both components $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\overline{W^2(\mathcal{A})}$ are connected disjoint compact subsets of $\mathbb{C}$. Hence there exists a piecewise smooth simply closed Jordan curve $\Gamma_1$ such that one of the components is located inside of $\Gamma_1$ while the other one is located outside of $\Gamma_1$ (see [LMMT01, Lemma 4.2]).

First we consider the case that $\mathcal{F}_1$ lies in the bounded component $\mathcal{U}_1$ of $\mathbb{C} \setminus \Gamma_1$. Then $\mathcal{U}_1$ is a bounded simply connected domain with

$$\mathcal{F}_1 \subset \mathcal{U}_1, \quad \mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$$

and the boundary $\Gamma_1$ of $\mathcal{U}_1$ is a piecewise smooth simply closed Jordan curve. We choose an element $f \in \mathcal{H}_1$ such that $Cf \neq 0$ and set

$$g(\lambda) := (D-\lambda)^{-1}Cf, \quad \lambda \in \overline{\mathcal{U}_1}.$$

For fixed $\lambda \in \overline{\mathcal{U}_1}$, we consider the function

$$\varphi_\lambda(\mu) := \Delta\big(f,g(\lambda);\mu\big), \quad \mu \in \mathbb{C}.$$

By Proposition 1.1.3, every zero of the quadratic polynomial $\varphi_\lambda$ lies in $W^2(\mathcal{A})$. Since $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$, it follows that $\varphi_\lambda$ has exactly one zero $\mu(\lambda) \in \mathcal{F}_1$. The function $\mu : \overline{\mathcal{U}_1} \to \mathcal{F}_1$ is continuous and maps $\overline{\mathcal{U}_1}$ into itself. Since $\mathcal{U}_1$ is simply connected and its boundary is a piecewise smooth simply closed Jordan curve, $\overline{\mathcal{U}_1}$ is homeomorphic to the closed unit disc (see *e.g.* [Hur64], Chapter III.6.4). Hence the Brouwer fixed point theorem (see *e.g.*

[DS88a, Sections V.10, V.12]) applies and yields that there exists at least one point $\lambda_1 \in \mathcal{F}_1$ such that $\mu(\lambda_1) = \lambda_1$. Then $\Delta\big(f, g(\lambda_1); \lambda_1\big) = 0$. On the other hand, (see (1.6.4)),

$$\begin{aligned}\Delta\big(f, g(\lambda_1); \lambda_1\big) &= \big(S_1(\lambda_1)f, f\big)\big(Cf, (D-\lambda_1)^{-1}Cf\big)\\ &= \big(S_1(\lambda_1)f, f\big)\big((D-\lambda_1)g(\lambda_1), g(\lambda_1)\big).\end{aligned} \qquad (1.6.6)$$

The second factor is non-zero since $g(\lambda_1) \neq 0$ and $\lambda_1 \notin W(D)$ (note that $W(D) \cap \mathcal{F}_1 = \emptyset$). Thus (1.6.6) implies $\big(S_1(\lambda_1)f, f\big) = 0$, that is, $\lambda_1 \in W(S_1)$.

Now consider the component $\mathcal{F}_2$ and the second Schur complement $S_2$. If there also exists a piecewise smooth simply closed Jordan curve $\Gamma_2$ such that $\mathcal{F}_2$ lies in the interior of $\Gamma_2$ and $\mathcal{F}_1$ in its exterior, then the proof of $W(S_2) \cap \mathcal{F}_2 \neq \emptyset$ is analogous to the above reasoning. Otherwise, we let $\mathcal{V}_1 := \overline{\mathbb{C}} \setminus \overline{\mathcal{U}_1}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, and we suppose for simplicity that $0 \in \mathcal{U}_1$. Since $B \neq 0$, there exists an element $g \in \mathcal{H}_2$ such that $Bg \neq 0$. Set

$$f(\lambda) := (A-\lambda)^{-1}Bg, \quad \lambda \in \mathbb{C} \setminus \mathcal{U}_1.$$

If $\lambda \in \mathbb{C} \setminus \mathcal{U}_1$ $\big(= \overline{\mathcal{V}_1} \setminus \{\infty\}\big)$ is fixed, the function

$$\psi_\lambda(\eta) := \Delta\big(f(\lambda), g; \eta\big), \quad \eta \in \mathbb{C},$$

has exactly one zero $\eta(\lambda) \in \mathcal{F}_2$. Evidently, the same holds for the function $\widetilde{\psi}_\lambda(\eta) := |\lambda|^2\psi_\lambda(\eta)$, $\eta \in \mathbb{C}$, which we can consider also for $\lambda = \infty$. More exactly, let $\widetilde{\psi}_\infty(\eta) := \lim_{\lambda\to\infty} \widetilde{\psi}_\lambda(\eta)$. Since $\lim_{\lambda\to\infty} \lambda(A-\lambda)^{-1} = -I$, it is easy to see that $\widetilde{\psi}_\infty(\eta) = \Delta(Bg, g; \eta)$. Hence the function $\widetilde{\psi}_\infty$ has exactly one zero $\eta(\infty) \in \mathcal{F}_2$. The function $\eta : \overline{\mathcal{V}_1} \to \mathcal{F}_2$ is continuous on $\overline{\mathcal{V}_1}$ and maps $\overline{\mathcal{V}_1}$ into itself. Since the boundary of $\mathcal{V}_1$ is a piecewise smooth simply closed Jordan curve, it follows that there exists at least one point $\lambda_2 \in \mathcal{F}_2$ such that $\eta(\lambda_2) = \lambda_2$, that is, $\Delta\big(f(\lambda_2), g; \lambda_2\big) = 0$. Using the equality

$$\Delta\big(f(\lambda_2), g; \lambda_2\big) = \big(S_2(\lambda_2)g, g\big)\big((A-\lambda_2)f(\lambda_2), f(\lambda_2)\big)$$

instead of (1.6.6), we obtain that $\big(S_2(\lambda_2)g, g\big) = 0$, that is, $\lambda_2 \in W(S_2)$.

In the second case that $\mathcal{F}_2$ lies in the bounded component of $\mathbb{C} \setminus \Gamma_1$, we consider the functions $f$ on $\overline{\mathcal{U}_1}$ and $g$ on $\mathbb{C} \setminus \mathcal{U}_1$ and proceed in the same way as above. □

**Remark 1.6.5** It is an open question whether the components of $\overline{W^2(\mathcal{A})}$ are simply connected, and, if no, one component may lie in a "hole" of the other one.

In the latter case, the second part of the above proof involving the construction with $\infty$ necessary. In fact, then there do not exist two piecewise smooth simply closed Jordan curves $\Gamma_i$, $i = 1, 2$, such that $\mathcal{F}_i$ lies in the interior of $\Gamma_i$ and in the exterior of the other curve. One of these curves, say $\Gamma_2$, can only be chosen to be a Cauchy contour; in fact, it may be chosen as the union of two piecewise smooth simply closed Jordan curves, one of them being the curve $\Gamma_1$ with opposite orientation, the other one being the positively oriented circle $\{z \in \mathbb{C} : |z| = R\}$ of radius $R > 0$ so that $\|\mathcal{A}\| < R$.

**Theorem 1.6.6** *Let* $\mathcal{H}_1$, $\mathcal{H}_2$ *be separable Hilbert spaces with* $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$. *Suppose that* $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ *consists of two components, enumerated so that* $W(S_1) \cap \mathcal{F}_1 \neq \emptyset$, $W(S_2) \cap \mathcal{F}_2 \neq \emptyset$. *Then:*

i) *for* $f \in \mathcal{H}_1$, $f \neq 0$, *the function* $\big(S_1(\cdot)f, f\big)$ *has exactly one zero in* $\mathcal{F}_1$, *for* $g \in \mathcal{H}_2$, $g \neq 0$, *the function* $\big(S_2(\cdot)g, g\big)$ *has exactly one zero in* $\mathcal{F}_2$;

ii) *the Schur complements* $S_1$ *and* $S_2$ *admit factorizations*

$$S_j(\lambda) = M_j(\lambda)(Z_j - \lambda), \quad \lambda \in \mathcal{F}_j, \tag{1.6.7}$$

*for* $j = 1, 2$ *where* $M_j : \mathcal{F}_j \to L(\mathcal{H}_j)$ *is an analytic operator function such that* $M_j(\lambda)$ *is boundedly invertible for all* $\lambda \in \mathcal{F}_j$ *and* $Z_j \in L(\mathcal{H}_j)$ *is such that* $\sigma(Z_j) \subset \mathcal{F}_j$.

***Proof.*** Let the bounded set $\mathcal{U}_1$ and the curve $\Gamma_1$ parametrizing its boundary be chosen as in the proof of Theorem 1.6.4 with $\mathcal{F}_1 \subset \mathcal{U}_1$, $\mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$ (the case $\mathcal{F}_2 \subset \mathcal{U}_1$, $\mathcal{F}_1 \cap \overline{\mathcal{U}_1} = \emptyset$ is treated analogously).

i) We prove the claim for $S_1$; the proof for $S_2$ is completely analogous.

First we assume that $\dim \mathcal{H}_1 =: n_1 < \infty$, $\dim \mathcal{H}_2 =: n_2 < \infty$. From [KMM93, Lemma 6] it follows that for arbitrary $f \in \mathcal{H}_1$, $f \neq 0$,

$$\operatorname{ind}_{\Gamma_1} \det\, S_1(\cdot) = n_1 \operatorname{ind}_{\Gamma_1} \big(S_1(\cdot)f, f\big) =: n_1 l_1, \tag{1.6.8}$$

where $\operatorname{ind}_{\Gamma_1} \big(S_1(\cdot)f, f\big)$ denotes the number $l_1$ of zeroes of $\big(S_1(\cdot)f, f\big)$ in $\mathcal{F}_1$. The factorization (1.6.2) implies that

$$\det(\mathcal{A} - \lambda) = \det S_1(\lambda)\, \det(D - \lambda), \quad \lambda \notin \sigma(D).$$

Since $\mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$ and $\sigma(D) \subset W(D) \subset \mathcal{F}_2$, the function $\lambda \mapsto \det(D - \lambda)$ does not have zeroes in the interior $\mathcal{U}_1$ of $\Gamma_1$, and hence

$$\operatorname{ind}_{\Gamma_1} \det(\mathcal{A} - \cdot) = \operatorname{ind}_{\Gamma_1} \det S_1(\cdot).$$

Therefore the operator $\mathcal{A}$ has exactly $n_1 l_1$ eigenvalues in $\mathcal{F}_1$, counted according to their algebraic multiplicities. Analogously, if for $g \in \mathcal{H}_2$, $g \neq 0$, we denote by $l_2$ the number of zeroes of the function $(S_2(\cdot)g, g)$ in the set $\mathcal{F}_2$, then the operator $\mathcal{A}$ has exactly $n_2 l_2$ eigenvalues in $\mathcal{F}_2$, again counted according to their algebraic multiplicities. Since the total number of eigenvalues of $\mathcal{A}$ is equal to $\dim \mathcal{H} = n_1 + n_2$, we obtain the equality

$$n_1 l_1 + n_2 l_2 = n_1 + n_2.$$

Due to Theorem 1.6.4, we have $l_1 \geq 1$, $l_2 \geq 1$ and hence $l_1 = l_2 = 1$. This proves i) in the finite-dimensional case.

In the general case, we choose sequences of orthogonal projections $P_n \in L(\mathcal{H}_1)$ and $Q_n \in L(\mathcal{H}_2)$, $n \in \mathbb{N}$, that converge strongly to the respective identity operators and have finite-dimensional ranges, $\dim R(P_n) \geq 1$, $\dim R(Q_n) \geq 1$. For $n \in \mathbb{N}$, we set

$$A_n := P_n A P_n, \quad B_n := P_n B Q_n, \quad C_n := Q_n C P_n, \quad D_n := Q_n D Q_n$$

and consider the matrix

$$\mathcal{A}_n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

in the space $R(P_n) \oplus R(Q_n)$. It is easy to see that, for all $n \in \mathbb{N}$,

$$W^2(\mathcal{A}_n) \subset W^2(\mathcal{A})$$

and that also $W^2(\mathcal{A}_n) = \overline{W^2(\mathcal{A}_n)}$ consists of two components, $W^2(\mathcal{A}_n) = \mathcal{F}_1^n \,\dot{\cup}\, \mathcal{F}_2^n$ with $W(A_n) \subset \mathcal{F}_1^n \subset \mathcal{F}_1$ and $W(D_n) \subset \mathcal{F}_2^n \subset \mathcal{F}_2$. Set

$$S_1^{(n)}(\lambda) := A_n - \lambda - B_n(D_n - \lambda)^{-1} C_n, \quad \lambda \notin \mathcal{F}_2.$$

If we prove that, for every $f \in \mathcal{H}_1$,

$$\sup_{\lambda \in \Gamma_1} \bigl| \bigl(S_1(\lambda) f, f\bigr) - \bigl(S_1^{(n)}(\lambda) f, f\bigr) \bigr| \longrightarrow 0, \quad n \to \infty, \tag{1.6.9}$$

then, according to what was shown in the first part of the proof for the finite-dimensional case,

$$\operatorname{ind}_{\Gamma_1}\bigl(S_1(\cdot) f, f\bigr) = \operatorname{ind}_{\Gamma_1}\bigl(S_1^{(n)}(\cdot) f, f\bigr) = 1$$

for every $f \in \mathcal{H}_1$, $f \neq 0$; this completes the proof of claim i) for $S_1$ in the general case.

In order to prove (1.6.9), let $f \in \mathcal{H}_1$ be arbitrary and write

$$\begin{aligned}
&\bigl(\bigl(S_1(\lambda) - S_1^{(n)}(\lambda)\bigr) f, f\bigr) \\
&= \bigl((A - A_n) f, f\bigr) - \bigl(B_n (D_n - \lambda)^{-1} (C - C_n) f, f\bigr) \\
&\quad - \bigl(B_n\bigl((D - \lambda)^{-1} - (D_n - \lambda)^{-1}\bigr) C f, f\bigr) - \bigl((B - B_n)(D - \lambda)^{-1} C f, f\bigr) \\
&= \bigl((A - A_n) f, f\bigr) - \bigl(B_n (D_n - \lambda)^{-1} (C - C_n) f, f\bigr) \\
&\quad - \bigl(B_n (D_n - \lambda)^{-1} (D_n - D)(D - \lambda)^{-1} C f, f\bigr) - \bigl((B - B_n)(D - \lambda)^{-1} C f, f\bigr).
\end{aligned}$$

Now $A_n \to A$ (strongly) for $n \to \infty$ implies that

$$\big((A-A_n)f,f\big)\longrightarrow 0,\quad n\to\infty.$$

Since $W(D_n)\subset W(D)\subset\mathcal{F}_2$ and $\mathcal{F}_2\cap\Gamma_1\subset\mathcal{F}_2\cap\overline{\mathcal{U}_1}=\emptyset$, we have

$$\big\|(D_n-\lambda)^{-1}\big\|,\ \big\|(D-\lambda)^{-1}\big\|\leq\frac{1}{\operatorname{dist}(\Gamma_1,\,\mathcal{F}_2)},\quad \lambda\in\Gamma_1,\ n\in\mathbb{N}.$$

Together with $\|B_n\|\leq\|B\|$ and $C_n\to C$ (strongly) for $n\to\infty$, we obtain

$$\sup_{\lambda\in\Gamma_1}\big|\big(B_n(D_n-\lambda)^{-1}(C-C_n)f,f\big)\big|\leq\|B\|\,\frac{1}{\operatorname{dist}(\Gamma_1,\,\mathcal{F}_2)}\big\|(C-C_n)f\big\|\,\|f\|$$
$$\longrightarrow 0,\quad n\to\infty.$$

Further, since also $B_n^*\to B^*$ (strongly) for $n\to\infty$, it follows that

$$\sup_{\lambda\in\Gamma_1}\big|\big((B-B_n)(D-\lambda)^{-1}Cf,f\big)\big|\leq\frac{1}{\operatorname{dist}(\Gamma_1,\,\mathcal{F}_2)}\,\|Cf\|\,\big\|(B^*-B_n^*)f\big\|$$
$$\longrightarrow 0,\quad n\to\infty.$$

Finally, the facts that $D_n\to D$ (strongly) for $n\to\infty$ and that the set $\big\{(D-\lambda)^{-1}Cf:\lambda\in\Gamma_1\big\}\subset\mathcal{H}_2$ is compact imply that

$$\sup_{\lambda\in\Gamma_1}\big\|(D_n-D)(D-\lambda)^{-1}Cf\big\|\longrightarrow 0,\quad n\to\infty.$$

Hence

$$\big\|B_n(D_n-\lambda)^{-1}(D_n-D)(D-\lambda)^{-1}Cf\big\|$$
$$\leq\|B\|\frac{1}{\operatorname{dist}(\Gamma_1,\,\mathcal{F}_2)}\big\|(D_n-D)(D-\lambda)^{-1}Cf\big\|\longrightarrow 0,\quad n\to\infty,$$

uniformly for $\lambda\in\Gamma_1$. This proves (1.6.9).

ii) Since $\mathcal{U}_1$ is simply connected, we obtain the factorization (1.6.7) for $S_1$ from the factorization theorem [MM75, Theorem 2, Remark 1)]. If also for $\mathcal{F}_2$ there exists a bounded simply connected domain $\mathcal{U}_2$ such that $\mathcal{F}_2\subset\mathcal{U}_2$, $\mathcal{F}_1\cap\overline{\mathcal{U}_2}=\emptyset$, the second relation in (1.6.7) follows from the same factorization theorem. If this is not the case, we consider the domain $\mathcal{U}_2:=\overline{\mathbb{C}}\setminus\overline{\mathcal{U}_1}$ which is simply connected in the extended plane $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Let $\mathcal{V}_2$ be the image of $\mathcal{U}_2$ under the inversion $\mu=\lambda^{-1}$ and set

$$W(\mu):=\mu\,S_2(\mu^{-1}),\quad \mu\in\mathcal{V}_2,\ \mu\neq 0. \tag{1.6.10}$$

If we define $W(0):=\lim_{\mu\to 0}W(\mu)=-I$, then $W$ is an analytic operator function in $\mathcal{V}_2$. It is easy to check that for $f\in\mathcal{H}_1$, $f\neq 0$, the function $\big(W(\cdot)f,f\big)$ has exactly one zero in $\mathcal{V}_2$. Therefore, by [MM75, Theorem 2, Remark 1)], the operator function $W$ admits a factorization

$$W(\mu)=Q(\mu)\,(Y-\mu),\quad \mu\in\mathcal{V}_2, \tag{1.6.11}$$

where $Q : \mathcal{V}_2 \to L(\mathcal{H}_2)$ is an analytic operator function on $\mathcal{V}_2$ with boundedly invertible values and $Y \in L(\mathcal{H}_2)$ with $\sigma(Y) \subset \mathcal{V}_2$. The operator $Y$ is invertible since $Q(0)\,Y = W(0) = -I$. Moreover, formula (1.6.11) implies that

$$S_2(\lambda) = \lambda\, W(\lambda^{-1}) = \lambda\, Q(\lambda^{-1})\,\bigl(Y - \lambda^{-1}\bigr) = -Q(\lambda^{-1})\,Y\,\bigl(Y^{-1} - \lambda\bigr).$$

If we set $M_2(\lambda) := -Q(\lambda^{-1})\,Y$, $\lambda \in \mathcal{U}_2$, and $Z_2 := Y^{-1}$, the factorization (1.6.7) for $S_2$ follows. □

**Remark 1.6.7** In Theorem 1.6.6 ii), the operator function $M_1$ is even analytic on $\rho(D)$ and $M_2$ is analytic on $\rho(A)$.
This follows by analytic continuation from the identity (1.6.7) since $S_1$ is analytic on $\rho(D)$ and $S_2$ is analytic on $\rho(A)$.

## 1.7 Block diagonalization

In this section we show that the quadratic numerical range yields a criterion for the block diagonalizability of a block operator matrix: If the closure of the quadratic numerical range consists of two components, then the block operator matrix can be transformed into diagonal form.

In order to prove this, we show that the two spectral subspaces corresponding to the two disjoint parts of the spectrum admit so-called angular operator representations. The diagonalizing matrix is constructed by means of the angular operators which, in addition, turn out to be solutions of Riccati equations associated with the block operator matrix.

In the following, a subset $\sigma \subset \sigma(\mathcal{A})$ is called an isolated part of $\sigma(\mathcal{A})$ if both $\sigma$ and $\sigma(\mathcal{A}) \setminus \sigma$ are closed. Associated with an isolated part $\sigma$ of $\sigma(\mathcal{A})$ is a *spectral subspace* $\mathcal{L}_\sigma$ which is defined as the range of the Riesz projection

$$P_\sigma := -\frac{1}{2\pi \mathrm{i}} \int_\Gamma (\mathcal{A} - z)^{-1} \mathrm{d}z$$

of $\mathcal{A}$ corresponding to $\sigma$; here $\Gamma$ is a Cauchy contour (that is, the finite union of simply closed rectifiable Jordan curves) such that $\sigma$ lies in its interior and $\sigma(\mathcal{A}) \setminus \sigma$ in its exterior. The spectral subspace $\mathcal{L}_\sigma$ is an invariant subspace of $\mathcal{A}$, *i.e.* $\mathcal{A}\mathcal{L}_\sigma \subset \mathcal{L}_\sigma$, and $\sigma\bigl(\mathcal{A}|_{\mathcal{L}_\sigma}\bigr) = \sigma$ (see *e.g.* [GGK90, Chapter I.2]).

**Theorem 1.7.1** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces with* $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$. *Suppose that $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \,\dot\cup\, \mathcal{F}_2$ consists of two components, enumerated such that $W(S_1) \cap \mathcal{F}_1 \neq \emptyset$, $W(S_2) \cap \mathcal{F}_2 \neq \emptyset$.*

*Then the spectrum $\sigma(\mathcal{A})$ separates into two parts, $\sigma(\mathcal{A}) = \sigma_1 \dot\cup \sigma_2$ with*

$$\sigma_1 := \sigma(\mathcal{A}) \cap \mathcal{F}_1 \neq \emptyset, \quad \sigma_2 := \sigma(\mathcal{A}) \cap \mathcal{F}_2 \neq \emptyset,$$

*and there exist bounded linear operators $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$, $K_2 \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that the following hold:*

i) *the spectral subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to $\sigma_1$ and $\sigma_2$, respectively, have angular operator representations*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x \\ K_1 x \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} K_2 y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\}; \tag{1.7.1}$$

ii) *the angular operators $K_1$ and $K_2$ satisfy the Riccati equations*

$$K_1 B K_1 + K_1 A - D K_1 - C = 0, \quad K_2 C K_2 + K_2 D - A K_2 - B = 0;$$

iii) *if $\Gamma_j$, $j = 1, 2$, is a Cauchy contour such that $\mathcal{F}_j$ lies in its interior and the other component $\overline{W^2(\mathcal{A})} \setminus \mathcal{F}_j$ in its exterior and if $Z_j$, $M_j$ are the operators and operator functions, respectively, in the factorization* (1.6.7) *of the Schur complement $S_j$, $j = 1, 2$, then*

$$K_1 = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D - \lambda)^{-1} C (Z_1 - \lambda)^{-1} \mathrm{d}\lambda, \tag{1.7.2}$$

$$K_2 = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2} (A - \lambda)^{-1} B (Z_2 - \lambda)^{-1} \mathrm{d}\lambda, \tag{1.7.3}$$

*and*

$$Z_1 = A + B K_1, \quad M_1(\lambda) = I - B(D - \lambda)^{-1} K_1, \quad \lambda \in \rho(D),$$
$$Z_2 = D + C K_2, \quad M_2(\lambda) = I - C(A - \lambda)^{-1} K_2, \quad \lambda \in \rho(A).$$

***Proof.*** Let $P$ and $Q$ be the Riesz projections of $\mathcal{A}$ corresponding to $\sigma_1$ and $\sigma_2$, respectively. With respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we write them as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

By [LMMT01, Lemma 4.2], at least one of the contours $\Gamma_1$ and $\Gamma_2$, say $\Gamma_1$, can be chosen to be a piecewise smooth simply closed Jordan curve. First we prove the statements for $K_1$.

i) The Frobenius-Schur factorization (1.6.2) implies that, for $\lambda \in \Gamma_1$,

$$(\mathcal{A} - \lambda)^{-1} \tag{1.7.4}$$
$$= \begin{pmatrix} S_1(\lambda)^{-1} & -S_1(\lambda)^{-1} B (D - \lambda)^{-1} \\ -(D - \lambda)^{-1} C S_1(\lambda)^{-1} & (D - \lambda)^{-1} + (D - \lambda)^{-1} C S_1(\lambda)^{-1} B (D - \lambda)^{-1} \end{pmatrix}.$$

As the resolvent of $D$ is holomorphic inside $\Gamma_1$, the matrix entries of the Riesz projection $P$ are given by

$$
\begin{aligned}
P_{11} &= -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} S_1(\lambda)^{-1} \mathrm{d}\lambda, \\
P_{12} &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} S_1(\lambda)^{-1} B(D-\lambda)^{-1} \mathrm{d}\lambda, \\
P_{21} &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C S_1(\lambda)^{-1} \mathrm{d}\lambda, \\
P_{22} &= -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C S_1(\lambda)^{-1} B(D-\lambda)^{-1} \mathrm{d}\lambda.
\end{aligned}
$$

We define the operator $K_1$ by (1.7.2). By [DK74a, Theorem I.3.2] (see also [GGK90, Theorem I.4.1]), it is the unique solution of the operator equation

$$K_1 Z_1 - D K_1 = C$$

with $Z_1 = A + BK_1$. By (1.6.7) and Remark 1.6.7, we have $S_1(\lambda)^{-1} = (Z_1-\lambda)^{-1} M_1(\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \rho(D)$. This and [DK74a, Lemma I.2.1] lead to

$$
\begin{aligned}
P_{22} &= -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C (Z_1-\lambda)^{-1} M_1(\lambda)^{-1} B (D-\lambda)^{-1} \mathrm{d}\lambda \\
&= \left( \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C (Z_1-\lambda)^{-1} \mathrm{d}\lambda \right) \\
&\qquad \cdot \left( \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (Z_1-\lambda)^{-1} M_1(\lambda)^{-1} B (D-\lambda)^{-1} \mathrm{d}\lambda \right) = K_1 P_{12}, \\
P_{21} &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C (Z_1-\lambda)^{-1} M_1(\lambda)^{-1} \mathrm{d}\lambda \\
&= \left( \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D-\lambda)^{-1} C (Z_1-\lambda)^{-1} \mathrm{d}\lambda \right) \\
&\qquad \cdot \left( -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (Z_1-\lambda)^{-1} M_1(\lambda)^{-1} \mathrm{d}\lambda \right) = K_1 P_{11}.
\end{aligned}
$$

These relations yield

$$P = \begin{pmatrix} P_{11} & P_{12} \\ K_1 P_{11} & K_1 P_{12} \end{pmatrix} = \begin{pmatrix} I \\ K_1 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \end{pmatrix}.$$

Since $\Gamma_1$ is a piecewise smooth simply closed Jordan curve, $P_{11}$ is bijective (see [MM75, Theorem 3]). Therefore the range $R\big((P_{11}\ P_{12})\big)$ is $\mathcal{H}_1$ and so

$$\mathcal{L}_1 = R(P) = R\left( \begin{pmatrix} I \\ K_1 \end{pmatrix} \right),$$

which proves the representation of $\mathcal{L}_1$ in (1.7.1).

ii) Let $x \in \mathcal{H}_1$ be arbitrary. Since the spectral subspace $\mathcal{L}_1$ is invariant for $\mathcal{A}$, there exists a $z \in \mathcal{H}_1$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ K_1 x \end{pmatrix} = \begin{pmatrix} z \\ K_1 z \end{pmatrix} \quad \Longleftrightarrow \quad \begin{aligned} Ax + BK_1 x &= z, \\ Cx + DK_1 x &= K_1 z. \end{aligned}$$

Inserting the first equation into the second, we find that

$$\big(C + DK_1 - K_1 A - K_1 B K_1\big)\, x = 0.$$

iii) By the definition of $K_1$ in (1.7.2), we have

$$\begin{aligned} BK_1 =& \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} B(D-\lambda)^{-1} C (Z_1 - \lambda)^{-1} \mathrm{d}\lambda \\ =& \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \big(A - \lambda - S_1(\lambda)\big)(Z_1 - \lambda)^{-1} \mathrm{d}\lambda \\ =& \frac{1}{2\pi \mathrm{i}} A \int_{\Gamma_1} (Z_1 - \lambda)^{-1} \mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \lambda (Z_1 - \lambda)^{-1} \mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} M_1(\lambda) \mathrm{d}\lambda \\ =& -A + Z_1; \end{aligned}$$

here we have used that, according to Theorem 1.6.6 ii), the spectrum of $Z_1$ lies in the interior of $\Gamma_1$ and that $M_1$ is analytic on $\rho(D)$ and hence in the interior of $\Gamma_1$ by Remark 1.6.7.

To prove the representation of $M_1(\lambda)$, we use the relation $Z_1 = A + BK_1$ and the Riccati equation for $K_1$ to obtain that, for $\lambda \in \rho(D)$,

$$\begin{aligned} &\big(I - B(D-\lambda)^{-1} K_1\big)\, (Z_1 - \lambda) \\ &= \big(I - B(D-\lambda)^{-1} K_1\big)\, (A + BK_1 - \lambda) \\ &= A - \lambda - B(D-\lambda)^{-1} \big(K_1(A + BK_1) - (D-\lambda)K_1 - \lambda K_1\big) \\ &= A - \lambda - B(D-\lambda)^{-1} C \\ &= S_1(\lambda). \end{aligned}$$

This completes the proof of all statements involving $K_1$.

The proofs of the statements for $K_2$ in i) to iii) are analogous if the contour $\Gamma_2$ can also be chosen to be a piecewise smooth simply closed Jordan curve.

Otherwise, for the Riesz projection $Q$ of $\mathcal{A}$ corresponding to $\sigma_2$, the bijectivity of $Q_{22}$ remains to be proved. In this case the contour $\Gamma_2$ can be chosen to consist of the two simply closed Jordan curves $-\Gamma_1$ (*i.e.* $\Gamma_1$ with opposite orientation) and the positively oriented circle $\{\lambda \in \mathbb{C} : |\lambda| = M\}$ where $M > 0$ is such that $\mathcal{F}_2$ is in its interior. The (piecewise smooth) contour $\Gamma_2$ is the boundary of a domain $\mathcal{U}_2$ such that $\mathcal{F}_2 \subset \mathcal{U}_2$. Without loss of generality, we assume that 0 lies inside $\Gamma_1$. Denote by $\Lambda_2$ the image of $\Gamma_2$ and by $\mathcal{V}_2$ the image of $\mathcal{U}_2$ under the inversion $\mu = \lambda^{-1}$. Then $\Lambda_2$ consists of the Jordan curves $\Lambda_2^+ = \{\mu \in \mathbb{C} : |\mu| = 1/M\}$ (inner boundary) and $\Lambda_2^-$ (outer boundary, the image of $-\Gamma_1$), and their orientation is again positive with respect to the image of $\mathcal{F}_2$. Then

$$Q_{22} = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2} S_2(\lambda)^{-1} \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_2} S_2(\mu^{-1})^{-1} \mu^{-2} \mathrm{d}\mu = \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_2} W(\mu)^{-1} \mu^{-1} \mathrm{d}\mu,$$

where $W(\mu) = \mu\, S_2(\mu^{-1})$ for $\mu \in \mathcal{V}_2$ as in (1.6.10). By (1.6.11),

$$\mu^{-1} W(\mu)^{-1} - Y^{-1} W(\mu)^{-1} = \mu^{-1} Y^{-1} Q(\mu)^{-1}.$$

The operator function on the right hand side is analytic on $\mathcal{V}_2$ since $0 \notin \mathcal{V}_2$. It follows that

$$Q_{22} = \frac{1}{2\pi \mathrm{i}} Y^{-1} \int_{\Lambda_2} W(\mu)^{-1} \mathrm{d}\mu = \frac{1}{2\pi \mathrm{i}} Y^{-1} \left( \int_{\Lambda_2^+} W(\mu)^{-1} \mathrm{d}\mu + \int_{\Lambda_2^-} W(\mu)^{-1} \mathrm{d}\mu \right).$$

The operator function $W(\cdot)^{-1}$ is analytic in the circle $\{\mu \in \mathbb{C} : |\mu| \le 1/M\}$, therefore the first integral equals 0. In the proof of Theorem 1.6.6 it was already used that the operator function $W$ fulfils the assumptions of [MM75, Theorem 2] with respect to $\Lambda_2^-$. By [MM75, Theorem 3], the operator

$$\frac{1}{2\pi \mathrm{i}} \int_{\Lambda_2^-} W(\mu)^{-1} \mathrm{d}\mu$$

is bijective and hence so is the operator $Q_{22}$. □

**Corollary 1.7.2** *Under the assumptions of Theorem* 1.7.1*, the block operator matrix $\mathcal{A}$ is similar to the block diagonal operator matrix*

$$\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \begin{pmatrix} A + BK_1 & 0 \\ 0 & D + CK_2 \end{pmatrix};$$

*in fact,*

$$\begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix} = \begin{pmatrix} A + BK_1 & 0 \\ 0 & D + CK_2 \end{pmatrix}.$$

***Proof.*** The operator $\begin{pmatrix} I \\ K_1 \end{pmatrix}$ maps $\mathcal{H}_1$ isomorphically on $\mathcal{L}_1$ and, by Theorem 1.7.1 ii) and iii),

$$\mathcal{A} \begin{pmatrix} I \\ K_1 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \end{pmatrix} Z_1, \quad \mathcal{A} \begin{pmatrix} K_2 \\ I \end{pmatrix} = \begin{pmatrix} K_2 \\ I \end{pmatrix} Z_2. \tag{1.7.5}$$

Therefore the restriction of the operator $\mathcal{A}$ to $\mathcal{L}_1$ is an operator which is similar to $Z_1$. Similarly, the restriction of the operator $\mathcal{A}$ to $\mathcal{L}_2$ is similar to $Z_2$. The second statement is immediate from (1.7.5). □

**Corollary 1.7.3** *If dim $\mathcal{H}_1 = n_1 < \infty$, then $\mathcal{F}_1$ contains exactly $n_1$ eigenvalues of $\mathcal{A}$ (counting multiplicities), and the first components of the corresponding eigenvectors and associated vectors form a basis in $\mathcal{H}_1$.*

**Corollary 1.7.4** *If $\mathcal{A} = \mathcal{A}^*$ in Theorem 1.7.1, then*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} -K^*y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\} \tag{1.7.6}$$

*with a uniform contraction $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ (i.e. $\|K\| < 1$).*

***Proof.*** Since $\mathcal{A}$ is self-adjoint, we have $\mathcal{L}_1 \perp \mathcal{L}_2$ and hence $K_2 = -K_1^*$. The strict inequality $\|K_1\| < 1$ cannot be proved with the methods used in this section; later, in Theorem 2.7.7, this is shown even for unbounded diagonal elements $A$, $D$ (see also [AL95, Lemma 2.2]). □

## 1.8 Spectral supporting subspaces

If a self-adjoint block operator matrix $\mathcal{A}$ has separated diagonal elements, $\sup W(D) < \alpha < \inf W(A)$ for some $\alpha \in \mathbb{R}$, then $\overline{W^2(\mathcal{A})}$ consists of two components. Then, by (1.7.1) (see also (1.7.6) above), the spectral subspace $\mathcal{L}_{(\alpha,\infty)}(\mathcal{A})$ is the graph of a bounded linear operator $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$.

In this section we drop the separation condition for the diagonal elements. We show that for intervals $\Delta \subset \rho(D)$, the corresponding spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ is the graph of a closed linear operator $K_1^\Delta : \mathcal{H}_1 \to \mathcal{H}_2$ which may only be defined on a subspace $\mathcal{H}_1^\Delta := \mathcal{D}(K_1^\Delta)$ of $\mathcal{H}_1$; the operator $K_1^\Delta$ is bounded if $\overline{\Delta} \subset \rho(D)$.

The main results of this section concern a description of the so-called spectral supporting subspace $\mathcal{H}_1^\Delta$ in terms of the Schur complement $S_1$. Analogous results hold for intervals $\Delta \subset \rho(A)$ and the corresponding spectral supporting subspaces $\mathcal{H}_2^\Delta$ (see [LMMT03]).

**Theorem 1.8.1** *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta \subset \mathbb{R}$ be an interval such that $\overline{\Delta} \subset \rho(D)$. Then there exists a subspace $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$ and a bounded linear operator $K_1^\Delta : \mathcal{H}_1^\Delta \to \mathcal{H}_2$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\}. \tag{1.8.1}$$

***Proof.*** A subspace $\mathcal{L} \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ is the graph of a linear operator if it contains no elements of the form $(0\ y)^{\mathrm{t}}$ with $y \neq 0$; it is the graph of a bounded linear operator if there is a $\gamma \geq 0$ with $\|y\| \leq \gamma \|x\|$ for all $(x\ y)^{\mathrm{t}} \in \mathcal{L}$.

Let $\alpha := \inf \Delta$, $\beta := \sup \Delta$ and set $\lambda_0 := (\alpha+\beta)/2$, $\delta := (\beta-\alpha)/2$. Then, for every $h = (x\ y)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$, we have $\|(\mathcal{A}-\lambda_0)h\| \le \delta\,\|h\|$. Thus

$$\|(D-\lambda_0)y + B^*x\| \le \|(\mathcal{A}-\lambda_0)h\| \le \delta\left(\|x\|^2+\|y\|^2\right)^{1/2} \le \delta\,(\|x\|+\|y\|).$$

Denote $\varrho := \operatorname{dist}\big(\lambda_0, \sigma(D)\big)$ $(>\delta)$. Then

$$\|(D-\lambda_0)y\| \ge \frac{1}{\big\|(D-\lambda_0)^{-1}\big\|}\|y\| \ge \varrho\,\|y\|,$$

and hence

$$\varrho\,\|y\| - \|B\|\,\|x\| \le \big\|(D-\lambda_0)y + B^*x\big\| \le \delta\,(\|x\|+\|y\|),$$

or $\|y\| \le \dfrac{\delta+\|B\|}{\varrho-\delta}\|x\|$. □

**Definition 1.8.2** If $\Delta$ is an interval as in Theorem 1.8.1, then $\mathcal{H}_1^\Delta$ is called the $\Delta$-*spectral supporting subspace* of $\mathcal{A}$ in $\mathcal{H}_1$.

Together with the analogue of Theorem 1.8.1 for intervals $\Delta \subset \mathbb{R}$ with $\overline{\Delta} \subset \rho(A)$, we obtain the following corollary.

**Corollary 1.8.3** *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta \subset \mathbb{R}$ be an interval such that $\overline{\Delta} \subset \rho(A) \cap \rho(D)$. Then there exist subspaces $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$, $\mathcal{H}_2^\Delta \subset \mathcal{H}_2$ and a bijective bounded linear operator $K_1^\Delta : \mathcal{H}_1^\Delta \to \mathcal{H}_2^\Delta$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} = \left\{ \begin{pmatrix} \big(K_1^\Delta\big)^{-1} y \\ y \end{pmatrix} : y \in \mathcal{H}_2^\Delta \right\}. \tag{1.8.2}$$

If we only assume that $\Delta \subset \rho(D)$, then the operator $K_1^\Delta$ is no longer bounded, but still closed:

**Theorem 1.8.4** *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta \subset \mathbb{R}$ be an open or half-open interval such that $\Delta \subset \rho(D)$. Then there exists a closed linear operator $K_1^\Delta : \mathcal{H}_1 \to \mathcal{H}_2$ with domain $\mathcal{H}_1^\Delta := \mathcal{D}(K_1^\Delta)$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\}. \tag{1.8.3}$$

***Proof.*** Let *e.g.* $\Delta = (\alpha,\beta) \subset \rho(D)$ with $\alpha \in \sigma(D)$. We use the same reasoning and notations as in the proof of Theorem 1.8.1. Assume that an element $h := (0\ \ y)^{\mathrm{t}}$ with $y \ne 0$ belongs to $\mathcal{L}_\Delta(\mathcal{A})$. Then we have $\|(D-\lambda_0)y\| = \|(\mathcal{A}-\lambda_0)h\| < \delta\,\|h\| = \delta\,\|y\|$ and, at the same time,

$$\|(D-\lambda_0)y\| \ge \frac{1}{\big\|(D-\lambda_0)^{-1}\big\|}\|y\| \ge \delta\,\|y\| > \|(D-\lambda_0)y\|,$$

a contradiction. This shows that $\mathcal{L}_\Delta(\mathcal{A})$ is the graph of a linear operator $K_1$. Since the subspace $\mathcal{L}_\Delta(\mathcal{A})$ is closed, $K_1$ is a closed operator. □

Note that $\mathcal{H}_1^\Delta$ is the first component of $\mathcal{L}_\Delta(\mathcal{A})$. The subspace $\mathcal{H}_1^\Delta$ is closed if $\overline{\Delta} \subset \rho(D)$; it is not necessarily closed if only $\Delta \subset \rho(D)$.

**Proposition 1.8.5** *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta = [\alpha, \beta] \subset \rho(D)$, $\gamma \in (\alpha, \beta)$, and $\Delta_1 := [\alpha, \gamma]$, $\Delta_2 := (\gamma, \beta]$. Then*

$$\mathcal{H}_1^\Delta = \mathcal{H}_1^{\Delta_1} \dot{+} \mathcal{H}_1^{\Delta_2};$$

*the subspaces $\mathcal{H}_1^{\Delta_1}$ and $\mathcal{H}_1^{\Delta_2}$ are orthogonal with respect to the inner product*

$$\langle x, y \rangle := \left( \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix}, \begin{pmatrix} y \\ K_1^\Delta y \end{pmatrix} \right), \quad x, y \in \mathcal{H}_1^\Delta,$$

*where $K_1^\Delta$ is the angular operator in the representation* (1.8.1) *of $\mathcal{L}_\Delta(\mathcal{A})$.*

***Proof.*** Since $\mathcal{A}$ is self-adjoint, we have

$$\mathcal{L}_\Delta(\mathcal{A}) = \mathcal{L}_{\Delta_1}(\mathcal{A}) \oplus \mathcal{L}_{\Delta_2}(\mathcal{A}).$$

On the other hand, by Theorem 1.8.1,

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\}, \quad \mathcal{L}_{\Delta_j}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^{\Delta_j} x \end{pmatrix} : x \in \mathcal{H}_1^{\Delta_j} \right\}, \ j = 1, 2,$$

with bounded linear operators $K_1^\Delta$, $K_1^{\Delta_1}$, $K_1^{\Delta_2} : \mathcal{H}_1 \to \mathcal{H}_2$. By projection onto the first component, $\mathcal{L}_\Delta(\mathcal{A})$ is mapped isomorphically onto $\mathcal{H}_1^\Delta$ and $\mathcal{L}_{\Delta_j}(\mathcal{A})$ isomorphically onto $\mathcal{H}_1^{\Delta_j}$, $j = 1, 2$. This implies the first statement. The second statement follows if we observe that $K_1^{\Delta_1}$, $K_1^{\Delta_2}$ are the restrictions of $K_1^\Delta$ to $\mathcal{H}_1^{\Delta_1}$ and $\mathcal{H}_1^{\Delta_2}$, respectively. □

**Proposition 1.8.6** *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta = [\alpha, \beta] \subset \rho(D)$ and denote by $\Omega$ the component of $\rho(D)$ containing $\Delta$. Let $\mathcal{H}_1^\Delta$ be the spectral supporting subspace corresponding to $\Delta$ and let $P_1^\Delta : \mathcal{H}_1 \to \mathcal{H}_1^\Delta$ be the projection of $\mathcal{H}_1$ onto $\mathcal{H}_1^\Delta$. Then the block operator matrix*

$$\mathcal{A}_\Delta := \begin{pmatrix} P_1^\Delta A P_1^\Delta & P_1^\Delta B \\ B^* P_1^\Delta & D \end{pmatrix} \tag{1.8.4}$$

*in $\mathcal{H}^\Delta := \mathcal{H}_1^\Delta \oplus \mathcal{H}_2$ satisfies*

i) *$\mathcal{L}_\Delta(\mathcal{A}_\Delta) = \mathcal{L}_\Delta(\mathcal{A})$;*

ii) *$\sigma(\mathcal{A}_\Delta) \cap \Omega \subset \Delta$.*

***Proof.*** i) The subspace $\mathcal{L}_\Delta(\mathcal{A}) \subset \mathcal{H}^\Delta$ is invariant under $\mathcal{A}$ and hence also under $\mathcal{A}_\Delta$. Moreover, $\mathcal{A}_\Delta|_{\mathcal{L}_\Delta(\mathcal{A})} = \mathcal{A}|_{\mathcal{L}_\Delta(\mathcal{A})}$ so that $\sigma\big(\mathcal{A}_\Delta|_{\mathcal{L}_\Delta(\mathcal{A})}\big) = \sigma\big(\mathcal{A}|_{\mathcal{L}_\Delta(\mathcal{A})}\big) \subset \Delta$. This shows that $\mathcal{L}_\Delta(\mathcal{A}) \subset \mathcal{L}_\Delta(\mathcal{A}_\Delta)$.

Using this inclusion and applying Theorem 1.8.1 to the operator $\mathcal{A}_\Delta$ in $\mathcal{H}^\Delta$, we find that there exists a subspace $\widetilde{\mathcal{H}}_1^\Delta \subset \mathcal{H}_1^\Delta$ and a bounded linear operator $\widetilde{K}_1^\Delta : \widetilde{\mathcal{H}}_1^\Delta \to \mathcal{H}_2$ so that

$$\mathcal{L}_\Delta(\mathcal{A}_\Delta) = \left\{ \begin{pmatrix} x \\ \widetilde{K}_1^\Delta x \end{pmatrix} : x \in \widetilde{\mathcal{H}}_1^\Delta \right\} \supset \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} = \mathcal{L}_\Delta(\mathcal{A}).$$

As a consequence, we obtain $\widetilde{\mathcal{H}}_1^\Delta = \mathcal{H}_1^\Delta$, $\widetilde{K}_1^\Delta = K_1^\Delta$, and hence the claim.

ii) It is sufficient to show that for every closed interval $\widetilde{\Delta}$ such that $\Delta \subset \widetilde{\Delta} \subset \Omega$, we have $\sigma(\mathcal{A}_\Delta) \cap \widetilde{\Delta} = \sigma(\mathcal{A}_\Delta) \cap \Delta$.

By i), $\mathcal{L}_\Delta(\mathcal{A}) = \mathcal{L}_\Delta(\mathcal{A}_\Delta) \subset \mathcal{L}_{\widetilde{\Delta}}(\mathcal{A}_\Delta)$. By Theorem 1.8.1 applied to $\mathcal{A}_\Delta$ in $\mathcal{H}^\Delta$ and the interval $\widetilde{\Delta}$, there exists a subspace $\widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} \subset \mathcal{H}_1^\Delta$ and a bounded linear operator $\widetilde{K}_1^{\widetilde{\Delta}} : \widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} \to \mathcal{H}_2$ such that

$$\mathcal{L}_{\widetilde{\Delta}}(\mathcal{A}_\Delta) = \left\{ \begin{pmatrix} x \\ \widetilde{K}_1^{\widetilde{\Delta}} x \end{pmatrix} : x \in \widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} \right\} \supset \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} = \mathcal{L}_\Delta(\mathcal{A}).$$

As in part i), we find $\widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} = \mathcal{H}_1^\Delta$, $\widetilde{K}_1^{\widetilde{\Delta}} = K_1^\Delta$, and hence $\mathcal{L}_{\widetilde{\Delta}}(\mathcal{A}_\Delta) = \mathcal{L}_\Delta(\mathcal{A}) = \mathcal{L}_\Delta(\mathcal{A}_\Delta)$. This proves $\sigma(\mathcal{A}_\Delta) \cap \widetilde{\Delta} = \sigma(\mathcal{A}_\Delta) \cap \Delta$. □

In the following we relate the spectral supporting subspace $\mathcal{H}_1^\Delta$ of the block operator matrix $\mathcal{A}$ to its Schur complement $S_1$. It turns out that $\mathcal{H}_1^\Delta$ is the maximal spectral subspace of $S_1$ corresponding to the interval $\Delta$.

**Theorem 1.8.7** *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta = [\alpha, \beta] \subset \rho(D)$ and let $\Gamma_\Delta$ be a simply closed Jordan curve surrounding $\Delta$ and intersecting $\mathbb{R}$ orthogonally in $\alpha$ and $\beta$. Define*

$$Q_\Delta := -\frac{1}{2\pi \mathrm{i}} \int'_{\Gamma_\Delta} S_1(z)^{-1} \,\mathrm{d}z, \tag{1.8.5}$$

*where $\int'$ denotes the Cauchy principal value at $\mathbb{R}$. Then the range of $Q_\Delta$ is given by $R\big(Q_\Delta\big) = \mathcal{H}_1^\Delta$.*

***Proof.*** Let $P_\Delta(\mathcal{A}) \in L(\mathcal{H})$ denote the orthogonal projection onto $\mathcal{L}_\Delta(\mathcal{A})$ and set $\Delta^\circ := (\alpha, \beta)$. We introduce the operator

$$\widehat{P}_\Delta(\mathcal{A}) := \frac{1}{2}\big(P_\Delta(\mathcal{A}) + P_{\Delta^\circ}(\mathcal{A})\big) = -\frac{1}{2\pi \mathrm{i}} \int'_{\Gamma_\Delta} (\mathcal{A} - z)^{-1} \,\mathrm{d}z.$$

Evidently, $R\big(\widehat{P}_\Delta(\mathcal{A})\big) = \mathcal{L}_\Delta(\mathcal{A})$. The inclusion $R\big(Q_\Delta\big) \subset \mathcal{H}_1^\Delta$ follows from

$$P_1 \widehat{P}_\Delta(\mathcal{A}) \begin{pmatrix} x \\ 0 \end{pmatrix} = Q_\Delta x, \quad x \in \mathcal{H}_1,$$

where $P_1 : \mathcal{H} \to \mathcal{H}_1$ is the projection of $\mathcal{H}$ onto its first component $\mathcal{H}_1$. In order to show that the range $R\big(Q_\Delta\big)$ is dense in $\mathcal{H}_1^\Delta$, consider $x_0 \in \mathcal{H}_1^\Delta$

with $x_0 \perp P_1 \widehat{P}_\Delta(\mathcal{A})\,(x\ 0)^{\mathrm{t}} = Q_\Delta x$, $x \in \mathcal{H}_1$. This implies

$$\left( \widehat{P}_\Delta(\mathcal{A}) \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \right) = 0, \quad x \in \mathcal{H}_1.$$

Since the projection $\widehat{P}_\Delta(\mathcal{A})$ is non-negative, we conclude $\widehat{P}_\Delta(\mathcal{A})(x_0\ 0)^{\mathrm{t}} = 0$. On the other hand, $\big(x_0\ K_1^\Delta x_0\big)^{\mathrm{t}} \in R\big(\widehat{P}_\Delta(\mathcal{A})\big)$ and thus

$$\|x_0\|^2 = \left( \begin{pmatrix} x_0 \\ K_1^\Delta x_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \right) = 0.$$

This shows that $R\big(Q_\Delta\big)$ is dense in $\mathcal{H}_1^\Delta$. The proof that $R\big(Q_\Delta\big) = \mathcal{H}_1^\Delta$ uses a similar reasoning: Suppose that $R\big(Q_\Delta\big)$ is not closed. Then there exists a sequence $(x_n)_1^\infty \subset \mathcal{H}_1^\Delta$, $\|x_n\| = 1$, such that $\|Q_\Delta x_n\| \to 0$ if $n \to \infty$. Hence

$$\left\| \widehat{P}_\Delta(\mathcal{A})^{1/2} \begin{pmatrix} x_n \\ 0 \end{pmatrix} \right\|^2 = \left( \widehat{P}_\Delta(\mathcal{A}) \begin{pmatrix} x_n \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ 0 \end{pmatrix} \right) = \left( P_1 \widehat{P}_\Delta(\mathcal{A}) \begin{pmatrix} x_n \\ 0 \end{pmatrix}, x_n \right)$$

$$= (Q_\Delta x_n, x_n) \longrightarrow 0, \quad n \to \infty.$$

Obviously, $\widehat{P}_\Delta(\mathcal{A})(x_n\ 0)^{\mathrm{t}} \to 0$ implies that $P_\Delta(\mathcal{A})(x_n\ 0)^{\mathrm{t}} \to 0$ for $n \to \infty$. Since $\big(x_n\ K_1^\Delta x_n\big)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$, we obtain the contradiction

$$1 = \|x_n\|^2 = \left( \begin{pmatrix} x_n \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ K_1^\Delta x_n \end{pmatrix} \right) = \left( \begin{pmatrix} x_n \\ 0 \end{pmatrix}, P_\Delta(\mathcal{A}) \begin{pmatrix} x_n \\ K_1^\Delta x_n \end{pmatrix} \right)$$

$$= \left( P_\Delta(\mathcal{A}) \begin{pmatrix} x_n \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ K_1^\Delta x_n \end{pmatrix} \right) \longrightarrow 0, \quad n \to \infty. \qquad \square$$

Next we describe a complementary subspace of the $\Delta$-spectral supporting subspace $\mathcal{H}_1^\Delta$ in $\mathcal{H}_1$ in terms of certain spectral subspaces of the Schur complement $S_1$.

**Theorem 1.8.8** *Let $\mathcal{A} = \mathcal{A}^*$ and $\Delta = [\alpha, \beta] \subset \rho(D)$.*

i) *If there exists a $\gamma > 0$ such that one of the conditions*

$$(\alpha, \alpha+\gamma) \cup (\beta-\gamma, \beta) \subset \rho(\mathcal{A}), \tag{1.8.6}$$

$$(\alpha-\gamma, \alpha) \cup (\beta, \beta+\gamma) \subset \rho(\mathcal{A}), \tag{1.8.7}$$

*is satisfied, then*

$$\mathcal{H}_1 = \overline{\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) \dotplus \mathcal{H}_1^\Delta \dotplus \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)}. \tag{1.8.8}$$

ii) *If, in addition, $\alpha$ and $\beta$ are isolated points of $\sigma(\mathcal{A})$ or in $\rho(\mathcal{A})$, then*

$$\mathcal{H}_1 = \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) \dotplus \mathcal{H}_1^\Delta \dotplus \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big) \tag{1.8.9}$$

*and hence*

$$\dim\,(\mathcal{H}_1 \ominus \mathcal{H}_1^\Delta) = \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) + \dim \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big). \tag{1.8.10}$$

**Remark 1.8.9** The dimension formula (1.8.10) can also be written as

$$\dim\big(\mathcal{H}_1 \ominus \mathcal{H}_1^{\Delta}\big) = \dim\big(S_1(\alpha)\big)_- + \dim\big(S_1(\beta)\big)_+,$$

where, for a bounded self-adjoint operator $T$, we denote by $T_\pm := \frac{1}{2}\big(T \pm |T|\big)$ the positive and negative part of $T$, respectively.
Moreover, since we have $\mathcal{H}_1^{\{\alpha\}} = \mathcal{L}_{\{0\}}\big(S_1(\alpha)\big) = \ker S_1(\alpha)$, we can replace $\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)$ by $\mathcal{L}_{(-\infty,0]}\big(S_1(\alpha)\big)$ in the relations (1.8.8) and (1.8.9) and, at the same time, $\mathcal{H}_1^{\Delta}$ by $\mathcal{H}_1^{(\alpha,\beta]}$. The same holds for the point $\beta$.

For the proof of Theorem 1.8.8, we provide a number of auxiliary lemmata concerning the monotonicity of the Schur complement $S_1$. In fact, $S_1$ is a monotonically decreasing operator function on $\rho(D) \cap \mathbb{R}$ with

$$S_1'(\lambda) = -I - B(D-\lambda)^{-2}B^* \le -I, \quad \lambda \in \rho(D) \cap \mathbb{R}, \tag{1.8.11}$$

and $\lim_{\lambda\to\infty} S_1(\lambda) = -\infty$. These observations are crucial in the following.

**Lemma 1.8.10** *Suppose that $\mathcal{A} = \mathcal{A}^*$ and let $\Delta = [\alpha, \beta] \subset \rho(D)$.*

i) *If $x = u + v$ where $u \in \mathcal{H}_1^{\Delta}$ and $\big(S_1(\beta)v, v\big) \ge 0$, then $\big(S_1(\alpha)x, x\big) \ge 0$.*
ii) *If $x = u + v$ where $u \in \mathcal{H}_1^{\Delta}$ and $\big(S_1(\alpha)v, v\big) \le 0$, then $\big(S_1(\beta)x, x\big) \le 0$.*

***Proof.*** We prove claim i); the proof of ii) is similar. Let $P_j : \mathcal{H} \to \mathcal{H}_j$ be the projection of $\mathcal{H}$ onto $\mathcal{H}_j$, $j = 1, 2$. For every $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{H}$, we have

$$\begin{aligned} S_1(\lambda)x &= (A-\lambda)x + By - B(D-\lambda)^{-1}B^*x - By \\ &= P_1(\mathcal{A}-\lambda)\mathbf{x} - B(D-\lambda)^{-1}\big(B^*x + (D-\lambda)y\big) \\ &= P_1(\mathcal{A}-\lambda)\mathbf{x} - B(D-\lambda)^{-1}P_2(\mathcal{A}-\lambda)\mathbf{x}. \end{aligned}$$

Let $\gamma := \max_{\lambda\in\Delta}\big(1 + \|B(D-\lambda)^{-1}\|\big) \le 1 + \|B\|\big(\mathrm{dist}\big(\Delta, \sigma(D)\big)\big)^{-1}$. Then

$$\|S_1(\lambda)x\| \le \big(1 + \|B(D-\lambda)^{-1}\|\big)\,\|(\mathcal{A}-\lambda)\mathbf{x}\| \ \le \gamma\,\|(\mathcal{A}-\lambda)\mathbf{x}\|. \tag{1.8.12}$$

Now let $x = u + v$ with $u \in \mathcal{H}_1^{\Delta}$ and $v \in \mathcal{H}_1$ such that $\big(S_1(\beta)v, v\big) \ge 0$. For arbitrary $n \in \mathbb{N}$, we decompose the interval $\Delta$ into $n$ subintervals

$$\begin{aligned} \Delta_k &:= \left[\alpha + \frac{(k-1)(\beta-\alpha)}{n},\ \alpha + \frac{k(\beta-\alpha)}{n}\right), \quad k = 1, 2, \dots, n-1, \\ \Delta_n &:= \left[\alpha + \frac{(n-1)(\beta-\alpha)}{n},\ \beta\right]. \end{aligned}$$

Let $E_{\mathcal{A}}(\Delta)$, $E_{\mathcal{A}}(\Delta_k)$ be the corresponding spectral projections of the operator $\mathcal{A}$. Then, for $\mathbf{u} = \big(u\ K_1^{\Delta}u\big)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$, we have $E_{\mathcal{A}}(\Delta)\mathbf{u} = \mathbf{u}$. If we set $\mathbf{u_k} := E_{\mathcal{A}}(\Delta_k)\mathbf{u}$ and $u_k := P_1\mathbf{u_k}$, $k = 1, \dots, n$, then $u = P_1\mathbf{u} = \sum_{k=1}^n u_k$. For $k = 1, 2, \dots, n$, we choose arbitrary points $\lambda_k \in \Delta_k$. By (1.8.12), we have

$$\big\|S_1(\lambda_k)u_k\big\| \le \gamma\,\big\|(\mathcal{A}-\lambda_k)\mathbf{u_k}\big\| \le \gamma\,\frac{\beta-\alpha}{n}\,\|\mathbf{u_k}\|.$$

Since $S_1$ is decreasing on $[\alpha,\beta]$ and all operators $S_1(\lambda)$ are self-adjoint for $\lambda\in\mathbb{R}$, it follows that

$$\begin{aligned}
\big(S_1(\alpha)x,x\big) &= \big(S_1(\alpha)(u+v),u+v\big) \ge \big(S_1(\lambda_1)(u+v),u+v\big)\\
&= \Big(S_1(\lambda_1)\Big(\sum_{k=2}^n u_k+v\Big),\sum_{k=2}^n u_k+v\Big) + \big(S_1(\lambda_1)u_1,u+v\big)\\
&\quad + \Big(\sum_{k=2}^n u_k+v, S_1(\lambda_1)u_1\Big)\\
&\ge \Big(S_1(\lambda_2)\Big(\sum_{k=2}^n u_k+v\Big),\sum_{k=2}^n u_k+v\Big) - \|S_1(\lambda_1)u_1\|\,\|u+v\|\\
&\quad - \Big\|\sum_{k=2}^n u_k+v\Big\|\,\big\|S_1(\lambda_1)u_1\big\|.
\end{aligned}$$

Since $\mathbf{u_1},\dots,\mathbf{u_n}$ are pairwise orthogonal, we have

$$\Big\|\sum_{j=k}^n u_j\Big\| \le \Big\|\sum_{j=k}^n \mathbf{u_j}\Big\| \le \|\mathbf{u}\|, \quad k=1,2,\dots,n.$$

Hence, if we let $\gamma' := \gamma\,(\|\mathbf{u}\|+\|v\|)$, then

$$\big(S_1(\alpha)x,x\big) \ge \Big(S_1(\lambda_2)\Big(\sum_{k=2}^n u_k+v\Big),\sum_{k=2}^n u_k+v\Big) - 2\gamma',\frac{\beta-\alpha}{n}\,\|\mathbf{u_1}\|.$$

Repeating these considerations, we finally obtain

$$\begin{aligned}
\big(S_1(\alpha)x,x\big) &\ge \Big(S_1(\lambda_3)\Big(\sum_{k=3}^n u_k+v\Big),\sum_{k=3}^n u_k+v\Big) - 2\gamma'\frac{\beta-\alpha}{n}\big(\|\mathbf{u_1}\|+\|\mathbf{u_2}\|\big)\\
&\ge \cdots\\
&\ge \big(S_1(\lambda_n)v,v\big) - 2\gamma'\,\frac{\beta-\alpha}{n}\sum_{k=1}^n\|\mathbf{u_k}\|\\
&\ge \big(S_1(\beta)v,v\big) - 2\gamma'\,\frac{\beta-\alpha}{n}\sum_{k=1}^n\|\mathbf{u_k}\|\\
&\ge -\,2\gamma'\,\frac{\beta-\alpha}{n}\,\sqrt{n}\,\Big(\sum_{k=1}^n\|\mathbf{u_k}\|^2\Big)^{1/2}\\
&= -2\gamma'\,\frac{\beta-\alpha}{\sqrt{n}}\,\|\mathbf{u}\|.
\end{aligned}$$

Since $n$ can be chosen arbitrarily large, it follows that $\big(S_1(\alpha)x,x\big)\ge 0$. $\square$

**Corollary 1.8.11** *Let $\mathcal{A} = \mathcal{A}^*$ and $\Delta = [\alpha, \beta] \subset \rho(D)$. Then*

$$\big(S_1(\alpha)x, x\big) \geq 0, \quad \big(S_1(\beta)x, x\big) \leq 0, \qquad x \in \mathcal{H}_1^\Delta;$$

*if $\Omega$ denotes the component of $\rho(D)$ containing $\Delta$, then*

$$\big(S_1(\lambda)x, x\big) \geq \ \ (\alpha - \lambda)\,\|x\|^2, \quad x \in \mathcal{H}_1^\Delta, \quad \text{if } \lambda \in \Omega,\ \lambda < \alpha,$$
$$\big(S_1(\lambda)x, x\big) \leq -(\lambda - \beta)\,\|x\|^2, \quad x \in \mathcal{H}_1^\Delta, \quad \text{if } \lambda \in \Omega,\ \lambda > \beta.$$

***Proof.*** The first two claims are immediate from Lemma 1.8.10 ii) if we set $v = 0$. The last claim follows from relation (1.8.11). □

**Corollary 1.8.12** *Let $\mathcal{A} = \mathcal{A}^*$ and $\Delta = [\alpha, \beta] \subset \rho(D)$. Then*

$$x \in \mathcal{H}_1^\Delta + \mathcal{L}_{[0,\infty)}\big(S_1(\beta)\big) \quad \Longrightarrow \quad \big(S_1(\alpha)x, x\big) \geq 0.$$

***Proof.*** Since $\big(S_1(\beta)v, v\big) \geq 0$ for $v \in \mathcal{L}_{[0,\infty)}\big(S_1(\beta)\big)$, the claim follows immediately from Lemma 1.8.10. □

The next lemma is a well-known fact about the spectra of positive perturbations of self-adjoint operators (see *e.g.* [BS87, (9.4.4)]).

**Lemma 1.8.13** *Let $T_1$, $T_2$ be bounded self-adjoint operators in a Hilbert space so that $(\mu, \nu) \subset \rho(T_1)$ and $\|T_2 - T_1\| < \nu - \mu$. Then*

$$T_1 \leq T_2 \quad \Longrightarrow \quad \big(\mu + \|T_2 - T_1\|, \nu\big) \subset \rho(T_2),$$
$$T_2 \leq T_1 \quad \Longrightarrow \quad \big(\mu, \nu - \|T_2 - T_1\|\big) \subset \rho(T_2).$$

**Lemma 1.8.14** *Suppose that $\mathcal{A} = \mathcal{A}^*$, $[\alpha, \alpha + \gamma) \subset \rho(D)$ for some $\gamma > 0$, and $0 \in \rho\big(S_1(\lambda)\big)$ for all $\lambda \in (\alpha, \alpha + \gamma)$. If, for such $\lambda$, we set*

$$a(\lambda) := \max\big(\sigma\big(S_1(\lambda)\big) \cap (-\infty, 0)\big), \quad b(\lambda) := \min\big(\sigma\big(S_1(\lambda)\big) \cap (0, \infty)\big),$$

*then $a$ and $b$ are continuous non-increasing functions on $(\alpha, \alpha + \gamma)$, and*

$$\big(0, b(\lambda_0)\big) \subset \rho\big(S_1(\alpha)\big), \quad \big(a(\lambda_0), 0\big) \subset \rho\big(S_1(\alpha + \gamma)\big), \quad \lambda_0 \in (\alpha, \alpha + \gamma).$$

***Proof.*** First suppose that $a(\lambda)$ and $b(\lambda)$ are finite for all $\lambda \in (\alpha, \alpha + \gamma)$. The continuity of $a$ and $b$ on $(\alpha, \alpha + \gamma)$ follows from [Kat95, Remark V.4.9] since $S_1$ is a continuous function of $\lambda$ with respect to the operator norm. In order to show that $b$ is non-increasing, it is sufficient to prove that for arbitrary $\lambda_0 \in (\alpha, \alpha + \gamma)$ there exists an $\varepsilon > 0$ such that

$$b(\lambda) \geq b(\lambda_0), \quad \lambda_0 - \varepsilon < \lambda < \lambda_0. \tag{1.8.13}$$

Choose $\varepsilon > 0$ such that $\|S_1(\lambda) - S_1(\lambda_0)\| < |a(\lambda_0)|$ for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$. Then, since $\big(a(\lambda_0), b(\lambda_0)\big) \subset \rho\big(S_1(\lambda_0)\big)$ and $S_1(\lambda) > S_1(\lambda_0)$, Lemma 1.8.13 implies that for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ we have

$$\big(0,b(\lambda_0)\big)\subset\big(a(\lambda_0)+\|S_1(\lambda)-S_1(\lambda_0)\|\,,b(\lambda_0)\big)\subset\rho\big(S_1(\lambda)\big),$$

and (1.8.13) follows.

In order to prove the last statement for $b$, let $\lambda_0\in(\alpha,\alpha+\gamma)$ be arbitrary and suppose that there exists a $\beta\in\big(0,b(\lambda_0)\big)$ such that $\beta\in\sigma\big(S_1(\alpha)\big)$. Then, again since $S_1$ is continuous in the operator norm, by [Kat95, Remark V.4.9] there exists a $\lambda'<\lambda_0$ in the neighbourhood of $\alpha$ such that

$$\big(0,b(\lambda_0)\big)\cap\sigma\big(S_1(\lambda')\big)\neq\emptyset.$$

Hence $b(\lambda')<b(\lambda_0)$, a contradiction to the fact that $b$ is non-increasing. The proofs for the function $a$ are analogous; it is also easy to check that all assertions remain true if $a$ or $b$ are no longer finite everywhere. □

**Lemma 1.8.15** *Let $\mathcal{A}=\mathcal{A}^*$ and $\alpha\in\rho(D)\cap\mathbb{R}$. Then*

$$(\alpha,\alpha+\gamma)\subset\rho(\mathcal{A})\text{ for some }\gamma>0\iff(0,\delta)\subset\rho\big(S_1(\alpha)\big)\text{ for some }\delta>0,$$

$$(\alpha-\gamma,\alpha)\subset\rho(\mathcal{A})\text{ for some }\gamma>0\iff(-\delta,0)\subset\rho\big(S_1(\alpha)\big)\text{ for some }\delta>0.$$

***Proof.*** We prove the first relation; the proof of the second one is analogous. If $(\alpha,\alpha+\gamma)\subset\rho(\mathcal{A})$ for some $\gamma>0$, then $0\in\rho\big(S_1(\lambda)\big)$ for all $\lambda\in(\alpha,\alpha+\gamma)$, and the assertion is immediate from Lemma 1.8.14. Conversely, if $(0,\delta)\subset\rho\big(S_1(\alpha)\big)$ for some $\delta>0$, then $(-\varepsilon,\delta-\varepsilon)\subset\rho\big(S_1(\alpha)-\varepsilon\big)$ for arbitrary $\varepsilon>0$. It is not difficult to see (*e.g.* using the resolvent identity for $D$) that $S_1(\alpha+\varepsilon)<S_1(\alpha)-\varepsilon$. Now choose $\varepsilon_0>0$ such that

$$\big\|S_1(\alpha)-\varepsilon-S_1(\alpha+\varepsilon)\big\|<\delta-\varepsilon,\quad 0<\varepsilon<\varepsilon_0.$$

Applying Lemma 1.8.13 (for the case $T_2<T_1$), we conclude that

$$\big(-\varepsilon,\delta-\varepsilon-\|S_1(\alpha)-\varepsilon-S_1(\alpha+\varepsilon)\|\big)\subset\rho\big(S_1(\alpha+\varepsilon)\big).$$

Thus $0\in\rho\big(S_1(\alpha+\varepsilon)\big)$, $0<\varepsilon<\varepsilon_0$, and hence $(\alpha,\alpha+\varepsilon_0)\subset\rho(\mathcal{A})$. □

**Lemma 1.8.16** *Let $T_n$, $n\in\mathbb{N}$, and $T$ be bounded self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $T_n$ is invertible, $\|T_n^{-1}\|\,\|T-T_n\|\le\omega$ for some $\omega>0$, $n\in\mathbb{N}$, and $\|T-T_n\|\to 0$, $n\to\infty$. If $x$, $\widehat{x}\in\mathcal{H}$ are such that $x=T\,\widehat{x}$, then*

$$\lim_{n\to\infty}\big(T_n^{-1}x,x\big)=(T\,\widehat{x},\widehat{x}).\tag{1.8.14}$$

***Proof.*** We have $T_n^{-1}x=T_n^{-1}T\,\widehat{x}=\widehat{x}+T_n^{-1}(T-T_n)\widehat{x}$ and hence

$$\|T_n^{-1}x\|\le(1+\omega)\,\|\widehat{x}\|,\quad n\in\mathbb{N}.\tag{1.8.15}$$

Further,

$$\big(T_n^{-1}x,x\big)=(\widehat{x},x)+\big(T_n^{-1}(T-T_n)\widehat{x},x\big)=(\widehat{x},T\,\widehat{x})+\big((T-T_n)\widehat{x},T_n^{-1}x\big),$$

and therefore, by (1.8.15),

$$\big|\big(T_n^{-1}x,x\big)-(\widehat{x},T\,\widehat{x})\big|\le\|T-T_n\|\,\|\widehat{x}\|\,\|T_n^{-1}x\|\le\|T-T_n\|\,(1+\omega)\,\|\widehat{x}\|.$$

Now (1.8.14) follows from the assumption $\|T-T_n\|\to 0$, $n\to\infty$. □

**Lemma 1.8.17** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspaces of a Hilbert space $\mathcal{H}$ and let $T$ be a bounded self-adjoint operator in $\mathcal{H}$. If*

$$(Tx,x)>0,\quad x\in\mathcal{L}_1,\ x\neq 0,\qquad (Ty,y)\le 0\quad y\in\mathcal{L}_2,\tag{1.8.16}$$

*then $\mathcal{L}_1\cap\mathcal{L}_2=\{0\}$. If the first inequality in (1.8.16) is sharpened to*

$$(Tx,x)\ge\delta\,\|x\|^2,\quad x\in\mathcal{L}_1,\tag{1.8.17}$$

*with some $\delta>0$, then $\mathcal{L}_1\dot{+}\mathcal{L}_2$ is closed.*

***Proof.*** The first assertion is obvious. In order to prove the second assertion, it is sufficient to show that there do not exist sequences $(x_n)_1^\infty\subset\mathcal{L}_1$ and $(y_n)_1^\infty\subset\mathcal{L}_2$ such that

$$\|x_n\|=1,\qquad \|x_n-y_n\|\longrightarrow 0,\quad n\to\infty,\tag{1.8.18}$$

(see *e.g.* [GK92, Theorem 2.1.1]). Assume to the contrary that such sequences do exist. Then the sequence $(y_n)_1^\infty$ is bounded and hence

$$\begin{aligned}\big|(Tx_n,x_n)-(Ty_n,y_n)\big|&\le\big|(Tx_n,x_n)-(Tx_n,y_n)\big|+\big|(Tx_n,y_n)-(Ty_n,y_n)\big|\\&\le\|T\|\,\|x_n\|\,\|x_n-y_n\|+\|T\|\,\|x_n-y_n\|\,\|y_n\|\\&\longrightarrow 0,\quad n\to\infty.\end{aligned}$$

Therefore $(Ty_n,y_n)\le 0$ implies $\limsup_{n\to\infty}(Tx_n,x_n)\le 0$, a contradiction to (1.8.17). □

Now we are ready for the proof of Theorem 1.8.8 on the description of complementary subspaces of spectral supporting subspaces.

***Proof of Theorem* 1.8.8.** i) Due to Lemma 1.8.15, the assumptions in (1.8.6) are equivalent to $(0,\delta)\subset\rho\big(S_1(\alpha)\big)$ and $(-\delta,0)\subset\rho\big(S_1(\beta)\big)$ for some $\delta>0$.

First we show that the sum in (1.8.8) is direct. By Corollary 1.8.11 and Lemma 1.8.17 (with $T=S_1(\beta)$), the sum $\mathcal{H}_1^\Delta+\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)$ is direct. From Corollary 1.8.12 and Lemma 1.8.17 it follows that also the sum $\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)+\big(\mathcal{H}_1^\Delta\dot{+}\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)\big)$ is direct.

In order to prove (1.8.8), assume to the contrary that an element $x_0\neq 0$ is orthogonal to the right hand side of (1.8.8). The relation $x_0\perp\mathcal{H}_1^{(\alpha,\beta)}$ implies that $(x_0\ 0)^{\mathrm{t}}\perp\mathcal{L}_{(\alpha,\beta)}(\mathcal{A})$, therefore the vector function

$(\mathcal{A}-z)^{-1}(x_0\ \ 0)^{\mathrm{t}}$, defined *e.g.* for all non-real $z$, has an analytic continuation onto the whole interval $(\alpha,\beta)$, and the same holds for the scalar function

$$\varphi(z) := \left((\mathcal{A}-z)^{-1}\begin{pmatrix}x_0\\0\end{pmatrix}, \begin{pmatrix}x_0\\0\end{pmatrix}\right) = \big(S_1(z)^{-1}x_0, x_0\big), \quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{1.8.19}$$

Obviously,

$$\varphi'(z) = \left((\mathcal{A}-z)^{-2}\begin{pmatrix}x_0\\0\end{pmatrix}, \begin{pmatrix}x_0\\0\end{pmatrix}\right), \quad z\in\mathbb{C}\setminus\mathbb{R}.$$

Hence $\varphi'(\lambda)>0$ for $\lambda\in(\alpha,\beta)$ so that $\varphi$ is increasing on $(\alpha,\beta)$.

First we prove (1.8.8) under the assumption (1.8.6). For every sequence $(\lambda_n)_1^\infty\subset(\alpha,\alpha+\gamma)$ such that $\lambda_n\searrow\alpha$, $n\to\infty$, the operators $T_n := S_1(\lambda_n)$ and $T := S_1(\alpha)$ satisfy the assumptions of Lemma 1.8.16. Indeed, since $S_1$ is continuous we have $\big\|S_1(\alpha)-S_1(\lambda_n)\big\|\to 0$, $n\to\infty$. Moreover, for $n\in\mathbb{N}$,

$$\begin{aligned}\big\|S_1(\lambda_n)^{-1}\big\| &\le \big\|(\mathcal{A}-\lambda_n)^{-1}\big\| \le (\lambda_n-\alpha)^{-1},\\ \big\|S_1(\lambda_n)-S_1(\alpha)\big\| &= \big\|\alpha-\lambda_n-(\alpha-\lambda_n)B(D-\lambda_n)^{-1}(D-\alpha)^{-1}B^*\big\|\\ &\le \big\|I+B(D-\lambda_n)^{-1}(D-\alpha)^{-1}B^*\big\|\,(\lambda_n-\alpha)\\ &\le \omega\,(\lambda_n-\alpha)\end{aligned}$$

with some constant $\omega>0$. Thus $\big\|S_1(\lambda_n)^{-1}\big\|\,\big\|S_1(\lambda_n)-S_1(\alpha)\big\|\le\omega$.

Now let $\widetilde{S} := S_1(\alpha)\big|_{\mathcal{L}_{(0,\infty)}(S_1(\alpha))}$. By definition, we have $\sigma\big(\widetilde{S}\big)\subset[0,\infty)$ and $\ker\widetilde{S}=\{0\}$. Since $(0,\delta)\subset\rho\big(S_1(\alpha)\big)$, we also have $(0,\delta)\subset\rho\big(\widetilde{S}\big)$; therefore, 0 is either a point of $\rho\big(\widetilde{S}\big)$ or an isolated point of $\sigma\big(\widetilde{S}\big)$. Because every isolated spectral point of a self-adjoint operator is an eigenvalue, the second case is excluded and so the operator $\widetilde{S}$ is boundedly invertible. From the assumption that $x_0\perp\mathcal{L}_{(-\infty,0]}\big(S_1(\alpha)\big)$ it follows that $x_0\in\mathcal{L}_{(0,\infty)}\big(S_1(\alpha)\big)$. If we set $\widehat{x}_0 := \widetilde{S}^{-1}x_0\in\mathcal{D}\big(\widetilde{S}\big)=\mathcal{L}_{(0,\infty)}\big(S_1(\alpha)\big)$, then $x_0=S_1(\alpha)\widehat{x}_0$ and, by Lemma 1.8.16,

$$\lim_{n\to\infty}\varphi(\lambda_n) = \lim_{n\to\infty}\big(S_1(\lambda_n)^{-1}x_0,x_0\big) = \big(S_1(\alpha)\,\widehat{x}_0,\widehat{x}_0\big)>0.$$

Analogously we prove that for every sequence $(\mu_n)_1^\infty\subset(\beta-\gamma,\beta)$ such that $\mu_n\nearrow\beta$, $n\to\infty$, there exists an $\widehat{x}_1\in\mathcal{L}_{(-\infty,0)}\big(S_1(\beta)\big)$ such that

$$\lim_{n\to\infty}\varphi(\mu_n) = \lim_{n\to\infty}\big(S_1(\mu_n)^{-1}x_0,x_0\big) = \big(S_1(\beta)\,\widehat{x}_1,\widehat{x}_1\big)<0.$$

Altogether, for sufficiently large $n\in\mathbb{N}$, we have $\lambda_n<\mu_n$, $\varphi(\lambda_n)>0$, and $\varphi(\mu_n)<0$, which contradicts the fact that $\varphi$ is increasing on $(\alpha,\beta)$.

The proof that (1.8.7) implies (1.8.8) is similar. Here we consider the subspace $\mathcal{R} := \mathcal{L}_{\mathbb{R}\setminus[\alpha,\beta]}(\mathcal{A}) = \mathcal{L}_{[\alpha,\beta]}(\mathcal{A})^\perp$ and the operator $\widetilde{\mathcal{A}} := \mathcal{A}|_{\mathcal{R}}$, for which $(\alpha-\gamma,\alpha)\cup(\beta,\beta+\gamma)\subset\rho\big(\widetilde{\mathcal{A}}\big)$. Then sequences $(\lambda_n)_1^\infty\subset(\alpha-\gamma,\alpha)$ and $(\mu_n)_1^\infty\subset(\beta,\beta+\gamma)$ with $\lambda_n\nearrow\alpha$ and $\mu_n\searrow\beta$, $n\to\infty$, are constructed leading to an analogous contradiction as above.

ii) It remains to be proved that the direct sum in (1.8.9) is closed. By Corollary 1.8.11,

$$\big(S_1(\beta)x,x\big)\leq 0,\quad x\in\mathcal{H}_1^\Delta.$$

The assumptions in ii) are equivalent to the fact that $R(\mathcal{A}-\alpha)$, $R(\mathcal{A}-\beta)$ are closed. From the Schur factorization (1.6.2) it is clear that this is equivalent to $R\big(S_1(\alpha)\big)$, $R\big(S_1(\beta)\big)$ being closed. Hence 0 is an isolated point of $\sigma\big(S_1(\alpha)\big)$ $\big(\sigma\big(S_1(\beta)\big)$, respectively$\big)$ or in $\rho\big(S_1(\alpha)\big)$ $\big(\rho\big(S_1(\beta)\big)$, respectively$\big)$. This implies that there exists a $\delta>0$ with

$$\big(S_1(\beta)x,x\big)\geq\delta\,\|x\|^2,\quad x\in\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big).$$

Then it follows from Lemma 1.8.17 ii) that the sum $\mathcal{H}_1^\Delta+\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)$ is closed. By Corollary 1.8.12,

$$\big(S_1(\alpha)x,x\big)\geq 0,\quad x\in\mathcal{H}_1^\Delta\dot{+}\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big).$$

As 0 is an isolated point of $\sigma\big(S_1(\alpha)\big)$ or in $\rho\big(S_1(\alpha)\big)$, there is a $\delta>0$ with

$$\big(S_1(\alpha)x,x\big)\leq-\delta\,\|x\|^2,\quad x\in\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big).$$

Again by Lemma 1.8.17, it follows that the sum in (1.8.9) is closed. □

**Corollary 1.8.18** *Let* $\alpha>\max\sigma(D)$. *Then*

$$\mathcal{H}_1=\overline{\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)\dot{+}\mathcal{H}_1^{[\alpha,\infty)}},\quad \mathcal{H}_1=\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)\dot{+}\mathcal{H}_1^{[\alpha,\infty)}.$$

***Proof.*** If we let $\beta>\max\sigma(\mathcal{A})$ arbitrarily large in Theorem 1.8.8, then $\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)=\{0\}$ in (1.8.8) and (1.8.9) since $\lim_{\lambda\to\infty}S_1(\lambda)=-\infty$. □

**Corollary 1.8.19** *If* $\Delta=[\alpha,\beta]\subset\rho(D)$, *then the following are equivalent:*

i) $\mathcal{H}_1^\Delta=\mathcal{H}_1$;

ii) $S_1(\alpha)\geq 0$, $S_1(\beta)\leq 0$.

***Proof.*** That i) implies ii) follows immediately from Corollary 1.8.11. Conversely, if ii) holds, then we choose two sequences $(\alpha_n)_1^\infty$, $(\beta_n)_1^\infty$ such that $\alpha_n\nearrow\alpha$, $\beta_n\searrow\beta$, $n\to\infty$, and $[\alpha_1,\beta_1]\subset\rho(D)$. According to (1.5.7), we have $S_1(\alpha_n)\gg 0$, $S_1(\beta_n)\ll 0$, $n\in\mathbb{N}$. Hence, by Theorem 1.8.8 or by [MS96] applied for every interval $\Delta_n:=[\alpha_n,\beta_n]$, we obtain $\mathcal{H}_1^{\Delta_n}=\mathcal{H}_1$. Obviously, $\mathcal{L}_\Delta(\mathcal{A})=\bigcap_{n=1}^\infty\mathcal{L}_{\Delta_n}(\mathcal{A})$, and hence

$$\mathcal{H}_1^\Delta=\bigcap_{n=1}^\infty\mathcal{H}_1^{\Delta_n}=\mathcal{H}_1.$$ □

**Remark 1.8.20** Certain analogues of the dimension formula (1.8.10), but not of the decomposition (1.8.9), were proved in the papers [MM87], [Kru93], [ADS00] for a rather general class of operator functions $S$, including even the non-analytic case. In the last two papers, the condition $S'(\lambda) \gg 0$ (or rather $S'(\lambda) \gg 0$ in [MM87]) was replaced by a weaker condition, called Virozub–Matsaev condition in [Kru93] and condition (**S**) in [ADS00]. In all three papers a strong additional assumption is imposed which implies the discreteness of the spectrum of $S$; as a consequence, in (1.8.10) instead of $\mathcal{H}_1^\Delta$ the closed span of all eigenvectors of $S$ to eigenvalues in $\Delta$ appears. It is easy to prove that, in the situation of [MM87] (or [Mar88, § 33]), also an analogue of (1.8.9) holds.

## 1.9 Variational principles for eigenvalues in gaps

The classical variational principles (see *e.g.* [WS72], [RS78]) provide a characterization of the eigenvalues of a semi-bounded self-adjoint operator $\mathcal{A}$ below or above its essential spectrum in terms of the *Rayleigh functional*

$$p(x) = \frac{(\mathcal{A}x, x)}{\|x\|^2}, \quad x \in \mathcal{D}(\mathcal{A}),\ x \neq 0.$$

For example, if $\lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of $\mathcal{A}$ below $\sigma_{\mathrm{ess}}(\mathcal{A})$, then

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathcal{D}(\mathcal{A}) \\ \dim \mathcal{L} = n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} p(x), \quad n = 1, 2, \ldots.$$

Note that the range of the Rayleigh functional is the numerical range of $\mathcal{A}$,

$$W(\mathcal{A}) = \big\{ p(x) : x \in \mathcal{D}(\mathcal{A}),\ x \neq 0 \big\}.$$

Even for bounded self-adjoint operators, eigenvalues in gaps of the essential spectrum cannot be characterized by such simple min-max principles. However, after suitable decomposition of the underlying Hilbert space, we may use that, for a self-adjoint block operator matrix $\mathcal{A}$, the quadratic numerical range is the union of the ranges $\Lambda_\pm(\mathcal{A})$ of the functionals $\lambda_\pm$:

$$W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \cup \Lambda_+(\mathcal{A})$$

where (see Corollary 1.1.4)

$$\lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \left( \frac{(Ax,x)}{\|x\|^2} + \frac{(Dy,y)}{\|y\|^2} \pm \sqrt{\left( \frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|y\|^2} \right)^2 + 4 \frac{|(By,x)|^2}{\|x\|^2\,\|y\|^2}} \right),$$

$$\Lambda_\pm(\mathcal{A}) = \left\{ \lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathcal{H}_1,\ y \in \mathcal{H}_2,\ x, y \neq 0 \right\}.$$

**Theorem 1.9.1** *Let $\mathcal{A} = \mathcal{A}^*$. Assume that there exists an $\alpha > \sup W(D)$ so that $\big(\sup W(D), \alpha\big) \subset \rho(\mathcal{A})$ and define*

$$\lambda_{\rm e} := \min\big(\sigma_{\rm ess}(\mathcal{A}) \cap (\sup W(D), \infty)\big). \tag{1.9.1}$$

*Further, let*

$$\kappa := \kappa_-(\lambda) := \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\lambda)\big) < \infty, \quad \lambda \in \big(\sup W(D), \alpha\big].$$

*If $\lambda_1 \le \lambda_2 \le \cdots$ is the finite or infinite sequence of the eigenvalues of $\mathcal{A}$ in the interval $\big(\sup W(D), \lambda_{\rm e}\big)$ counted with multiplicities, then*

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=\kappa+n}} \max_{\substack{x\in\mathcal{L}\\ x\neq 0}} \max_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \quad n = 1, 2, \dots . \tag{1.9.2}$$

***Proof.*** First we observe that the index shift $\kappa$ in (1.9.2) is independent of $\lambda$. In fact, by means of continuity arguments, it can be shown that $\kappa_-(\cdot)$ is constant on each subinterval of $\rho(S_1)$ (see [EL04]). Define

$$\mu_{n+\kappa} := \inf_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=\kappa+n}} \sup_{\substack{x\in\mathcal{L}\\ x\neq 0}} \sup_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \quad n = 1, 2, \dots .$$

Let $\alpha' \in \big(\sup W(D), \alpha\big) \subset \rho(\mathcal{A})$. Then $\alpha' \in \rho(S_1)$ by Proposition 1.6.2. First we prove that $\lambda_n \le \mu_{\kappa+n}$. To this end, set $\Delta := [\lambda_n, \infty)$. By Theorem 1.8.1, the spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ admits the representation

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x\\ K_1^\Delta x\end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} \tag{1.9.3}$$

where $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$ is a subspace and $K_1^\Delta : \mathcal{H}_1^\Delta \to \mathcal{H}_2$ is a bounded linear operator. According to Corollary 1.8.18, we have the decompositions

$$\mathcal{H}_1 = \mathcal{H}_1^{[\alpha',\infty)} \dot{+} \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha')\big) = \mathcal{H}_1^\Delta \dot{+} \mathcal{L}_{(-\infty,0)}\big(S_1(\lambda_n)\big). \tag{1.9.4}$$

Since $\big(\sup W(D), \alpha\big) \subset \rho(\mathcal{A})$ and $\alpha' < \alpha$, we have $\Delta = [\lambda_n, \infty) \subset [\alpha', \infty)$ and hence $\mathcal{H}_1^\Delta \subset \mathcal{H}_1^{[\alpha',\infty)}$. By (1.9.4) and by definition of $\kappa$, we obtain

$$\operatorname{codim}_{\mathcal{H}_1} \mathcal{H}_1^\Delta = \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha')\big) + \operatorname{codim}_{\mathcal{H}_1^{[\alpha',\infty)}} \mathcal{H}_1^\Delta = \kappa + n - 1.$$

Now let $\mathcal{L} \subset \mathcal{H}_1$ be an arbitrary subspace with $\dim \mathcal{L} = \kappa + n$. Then $\dim \mathcal{L} > \operatorname{codim}_{\mathcal{H}_1} \mathcal{H}_1^\Delta$ and hence there exists an $x \in \mathcal{L} \cap \mathcal{H}_1^\Delta$, $x \neq 0$. If $K_1^\Delta x \neq 0$, then (1.9.3) implies that

$$\begin{aligned} \lambda_n &\le \frac{1}{\|x\|^2 + \|K_1^\Delta x\|^2} \left( \mathcal{A}\begin{pmatrix} x\\ K_1^\Delta x\end{pmatrix}, \begin{pmatrix} x\\ K_1^\Delta x\end{pmatrix}\right) \\ &= \frac{1}{\|x\|^2 + \|K_1^\Delta x\|^2} \left( \mathcal{A}_{x,K_1^\Delta x}\begin{pmatrix} \|x\|\\ \|K_1^\Delta x\|\end{pmatrix}, \begin{pmatrix} \|x\|\\ \|K_1^\Delta x\|\end{pmatrix}\right)_{\mathbb{C}^2} \le \lambda_+\begin{pmatrix} x\\ K_1^\Delta x\end{pmatrix}; \end{aligned}$$

for the last inequality we have used that $\lambda_+(x, K_1^\Delta x)$ is the larger of the two eigenvalues of $\mathcal{A}_{x,K_1^\Delta x}$. If $K_1^\Delta x = 0$, then (1.2.6) shows that for every $y \in \mathcal{H}_2$, $y \neq 0$,

$$\lambda_n \leq \frac{1}{\|x\|^2 + \|K_1^\Delta x\|^2} \left( \mathcal{A} \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix}, \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} \right) = \frac{(Ax, x)}{\|x\|^2} \leq \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}.$$

So in each case we have found elements $x \in \mathcal{L}$, $y \in \mathcal{H}_2$ with $\lambda_n \leq \lambda_+(x, y)$ and hence $\lambda_n \leq \mu_{\kappa+n}$ .

Next we prove that $\lambda_n \geq \mu_{\kappa+n}$. In the same way as above, we see $\lambda_n \geq \mu_{\kappa+n}$ that

$$\mathcal{H}_1 = \mathcal{H}_1^{[\alpha',\infty)} \dot{+} \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha')\big) = \mathcal{H}_1^{(\lambda_n,\infty)} \dot{+} \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big)$$

and

$$\dim \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big) = \kappa + n.$$

Now choose $\mathcal{L} = \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big)$. If $x \in \mathcal{L}$, $y \in \mathcal{H}_2$, $x, y \neq 0$, and $\xi$, $\eta \in \mathbb{C}$ are arbitrary and we set $\mathbf{u} := (u \ v)^{\mathrm{t}} := \big(\xi\, x/\|x\| \ \eta\, y/\|y\|\big)^{\mathrm{t}}$, then we have $u \in \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big)$ and

$$\frac{1}{|\xi|^2 + |\eta|^2} \left( \mathcal{A}_{x,y} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)_{\mathbb{C}^2} = \frac{(\mathcal{A}\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|^2}.$$

Using the Frobenius-Schur factorization (1.6.2) of $\mathcal{A} - \lambda_n$, we obtain

$$\begin{aligned} &\left( \big(\mathcal{A} - \lambda_n\big) \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) \\ &= \left( \begin{pmatrix} S_1(\lambda_n) & 0 \\ 0 & D - \lambda_n \end{pmatrix} \begin{pmatrix} I & 0 \\ (D-\lambda_n)^{-1}B^* & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} I & 0 \\ (D-\lambda_n)^{-1}B^* & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right) \\ &= \big(S_1(\lambda_n)u, u\big) + \big((D - \lambda_n)w, w\big) \leq 0 \end{aligned}$$

where $w := (D - \lambda_n)^{-1} B^* u + v$. This implies $W(\mathcal{A}_{x,y}) \subset (-\infty, \lambda_n]$ and hence $\lambda_+(x, y) \leq \lambda_n$. Thus the proof of $\lambda_n = \mu_{\kappa+n}$ is complete.

It remains to be shown that the infimum and the suprema in $\mu_{\kappa+n}$ are all attained. Since $\lambda_n \in \sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A})$, there exists an eigenvector $(x_n \ y_n)^{\mathrm{t}} = (x_n \ K_1^\Delta x_n)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$ such that $\lambda_n = \lambda_+(\widehat{x}_n, \widehat{y}_n)$ where $\widehat{x}_n$, $\widehat{y}_n$ are such that $x_n = \|x_n\|\widehat{x}_n$, $y_n = \|y_n\|\widehat{y}_n$. □

We will further extend on variational principles for eigenvalues in gaps in Sections 2.10 and 2.11. There we generalize the above result to unbounded block operator matrices with real quadratic numerical range, including self-adjoint and certain $\mathcal{J}$-self-adjoint block operator matrices. The proof in the unbounded case uses variational principles for the Schur complements. We also present a method to calculate the index shift and establish two-sided eigenvalue estimates in terms of the entries of the block operator matrix. The results in Sections 2.10 and 2.11 may easily be specialized to bounded block operator matrices by observing that in this case $\mathcal{D}(A) = \mathcal{D}(B^*) = \mathcal{H}_1$ and $\mathcal{D}(B) = \mathcal{D}(D) = \mathcal{H}_2$.

## 1.10 $\mathcal{J}$-self-adjoint block operator matrices

$\mathcal{J}$-self-adjoint block operator matrices arise naturally when studying self-adjoint operators in Krein spaces. In fact, a Krein space is an inner product space $(\mathcal{K}, [\cdot,\cdot])$ that admits a decomposition $\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-$ so that $\mathcal{H}_1 = (\mathcal{K}_+, [\cdot,\cdot])$ and $\mathcal{H}_2 = (\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces (see [Lan82, Section I.1]). With respect to the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, we have $[\cdot,\cdot] = (\mathcal{J}\cdot,\cdot)$ with $\mathcal{J} = \operatorname{diag}(I, -I)$ as in Definition 1.1.14 and every bounded self-adjoint operator in $\mathcal{K}$ has a block operator matrix representation (1.1.3) with $A = A^*$, $D = D^*$, and $C = -B^*$, *i.e.* $\mathcal{A}$ is $\mathcal{J}$-self-adjoint.

In the following, we use the block operator techniques developed in the previous sections to study the spectrum of $\mathcal{J}$-self-adjoint block operator matrices; in particular, we identify intervals on which the spectrum is of definite type; on such intervals, the operator possesses a local spectral function. We also study the corresponding spectral supporting subspaces, thus establishing results analogous to those derived in Section 1.8 for the self-adjoint case.

We start with some elementary properties of the quadratic numerical range of $\mathcal{J}$-self-adjoint block operator matrices. Here an important role is played by the sets $\Lambda_\pm(\mathcal{A})$ introduced in Corollary 1.1.4; in the particular case $C = -B^*$, we have

$$\operatorname{dis}_{\mathcal{A}}(x,y) = \left(\frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|g\|^2}\right)^2 - 4\frac{|(By,x)|^2}{\|x\|^2\,\|y\|^2}$$

for $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, and, if $\operatorname{dis}_{\mathcal{A}}(x,y) \geq 0$,

$$\lambda_\pm\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\left(\frac{(Ax,x)}{\|x\|^2} + \frac{(Dy,y)}{\|y\|^2} \pm \sqrt{\left(\frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|y\|^2}\right)^2 - 4\frac{|(By,x)|^2}{\|x\|^2\,\|y\|^2}}\right),$$

$$\Lambda_\pm := \Lambda_\pm(\mathcal{A}) = \left\{\lambda_\pm\begin{pmatrix}x\\y\end{pmatrix} : x \in \mathcal{H}_1,\ y \in \mathcal{H}_2,\ x, y \neq 0,\ \operatorname{dis}_{\mathcal{A}}(x,y) \geq 0\right\}.$$

If $C = -B^*$ and $W^2(\mathcal{A})$ is real, then we have $\operatorname{dis}_{\mathcal{A}}(x,y) \geq 0$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, and hence $W^2(\mathcal{A}) = \Lambda_- \cup \Lambda_+$. In this case, continuity arguments show that the sets $\Lambda_\pm$ are intervals. In general, we can only prove a weaker statement (see Proposition 1.10.3 ii) below).

**Lemma 1.10.1** *For all* $(x_1\ y_1)^{\rm t}, (x_2\ y_2)^{\rm t} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $x_1, x_2, y_1, y_2 \neq 0$, *there exists a curve* $\big(x(t)\ y(t)\big)^{\rm t}$, $t \in [0,1]$, *such that*

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \qquad x(t),\, y(t) \neq 0, \quad t \in [0,1].$$

***Proof.*** There exist at most two different points $\zeta \in \mathbb{C}$, say $\zeta_1, \zeta_2$, such that $\zeta x_1 + (1-\zeta)x_2 = 0$ or $\zeta y_1 + (1-\zeta) y_2 = 0$. Let $\zeta(t)$, $t \in [0,1]$, be a curve in the complex plane with $\zeta(0) = 0$, $\zeta(1) = 1$ not passing through $\zeta_1$ and $\zeta_2$ . Then a curve with the required properties is given by

$$\zeta(t)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \big(1 - \zeta(t)\big)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad t \in [0,1]. \qquad \square$$

In the following we always assume that $\dim \mathcal{H} > 2$; otherwise, $\mathcal{A}$ is a $2 \times 2$ matrix and the questions considered here are trivial.

**Proposition 1.10.2** *Suppose* $\dim \mathcal{H} > 2$ *and let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint.*

i) *If* $W^2(\mathcal{A}) \setminus \mathbb{R} \neq \emptyset$, *then* $W^2(\mathcal{A})$ *is connected.*
ii) *If* $\overline{W^2(\mathcal{A})}$ *consists of two components, then* $\sigma(\mathcal{A}) \subset \mathbb{R}$.

***Proof.*** i) Let $z_1 \in W^2(\mathcal{A}) \setminus \mathbb{R}$. Then $z_1 \in \sigma_{\rm p}(\mathcal{A}_{x_1,y_1})$ for some $x_1 \in \mathcal{H}_1$, $y_1 \in \mathcal{H}_2$, $x_1, y_1 \neq 0$, with $\operatorname{dis}_{\mathcal{A}}(x_1, y_1) < 0$. Since $\dim \mathcal{H} > 2$, we either have $\dim \mathcal{H}_1 \geq 2$ or $\dim \mathcal{H}_2 \geq 2$ and thus, by Theorem 1.1.9, either $W(D) \subset W^2(\mathcal{A})$ or $W(A) \subset W^2(\mathcal{A})$. Because $A$ and $D$ are self-adjoint, this implies that there exists a $z_2 \in W^2(\mathcal{A}) \cap \mathbb{R}$. Hence $z_2 \in \sigma_{\rm p}(\mathcal{A}_{x_2,y_2})$ for some $x_2 \in \mathcal{H}_1$, $y_2 \in \mathcal{H}_2$, $x_2, y_2 \neq 0$ with $\operatorname{dis}_{\mathcal{A}}(x_2, y_2) \geq 0$.

Let $\big(x(t)\ y(t)\big)^{\rm t}$, $t \in [0,1]$, be a curve in $\mathcal{H}$ as in Lemma 1.10.1 connecting $(x_1\ y_1)^{\rm t}$ and $(x_2\ y_2)^{\rm t}$. Since $t \mapsto \operatorname{dis}_{\mathcal{A}}\big(x(t), y(t)\big)$ is continuous, there exists a $t_0 \in [0,1]$ such that $\operatorname{dis}_{\mathcal{A}}\big(x(t_0), y(t_0)\big) = 0$. This means that the matrix $\mathcal{A}_{x(t_0),y(t_0)}$ has a double eigenvalue, which we denote by $z_0$.

Now let $z, z' \in W^2(\mathcal{A})$ be arbitrary, $z \in \sigma_{\rm p}(\mathcal{A}_{x,y})$, $z' \in \sigma_{\rm p}(\mathcal{A}_{x',y'})$ for some $x, x' \in \mathcal{H}_1$, $y, y' \in \mathcal{H}_2$, $x, x', y, y' \neq 0$. By Lemma 1.10.1, there exists a curve $\big(x(t)\, y(t)\big)^{\rm t}$, $t \in [0,1]$, in $\mathcal{H}$ connecting $(x\, y)^{\rm t}$ with $\big(x(t_0)\ y(t_0)\big)^{\rm t}$. If $z = \lambda_\pm(x,y)$, then $\lambda_\pm\big(x(t), y(t)\big)$, $t \in [0,1]$, is a curve in $W^2(\mathcal{A})$ from $z$ to $z_0$. A curve from $z_0$ to $z'$ in $W^2(\mathcal{A})$ is constructed analogously.

ii) Since $W^2(\mathcal{A})$ consists of at most two components and $\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$, the claim is immediate from i). $\square$

In the following, we investigate the sets $\Lambda_\pm$ in greater detail. As in Proposition 1.3.9, we use the notations

$$a_- := \inf W(A), \quad a_+ := \sup W(A),$$
$$d_- := \inf W(D), \quad d_+ := \sup W(D).$$

**Proposition 1.10.3** *Let $\mathcal{A}$ be $\mathcal{J}$-self-adjoint.*

i) *Then* $\overline{\operatorname{conv}(\Lambda_- \cup \Lambda_+)} \subset \overline{\operatorname{conv}\big(W(A) \cup W(D)\big)}$ *or, equivalently,*

$$\inf \Lambda_- \geq \min\{a_-, d_-\}, \quad \sup \Lambda_+ \leq \max\{a_+, d_+\},$$

*and equality holds if* $\dim \mathcal{H}_1 \geq 2$ *and* $\dim \mathcal{H}_2 \geq 2$.

ii) *If* $\inf \Lambda_+ < \sup \Lambda_-$, *then*

$$(\inf \Lambda_-, \inf \Lambda_+] \subset \Lambda_-, \quad [\sup \Lambda_-, \sup \Lambda_+) \subset \Lambda_+.$$

***Proof.*** We prove the second statements in i) and ii); the proof of the first claims is similar.

i) For all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, with $\operatorname{dis}_{\mathcal{A}}(x,y) \geq 0$, we have

$$\begin{aligned}
\lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} &\leq \frac{1}{2}\left( \frac{(Ax,x)}{\|x\|^2} + \frac{(Dy,y)}{\|y\|^2} + \left| \frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|y\|^2} \right| \right) \\
&= \max\left\{ \frac{(Ax,x)}{\|x\|^2}, \frac{(Dy,y)}{\|y\|^2} \right\} \leq \max\{a_+, d_+\}
\end{aligned}$$

and thus $\sup \Lambda_+ \leq \max\{a_+, d_+\}$. Since $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$, we have $W(A) \subset W^2(\mathcal{A})$, $W(D) \subset W^2(\mathcal{A})$ by Theorem 1.1.9, and so equality follows.

ii) Assume to the contrary that there exists a $\lambda_0 \in [\sup \Lambda_-, \sup \Lambda_+)$ with $\lambda_0 \notin \Lambda_+$. Since $\lambda_0 < \sup \Lambda_+$, there is a $\lambda_1 \in \Lambda_+$ with $\lambda_0 < \lambda_1$. By assumption, there exists $\lambda_2 \in \Lambda_- \cap \Lambda_+$. Then $\lambda_2 < \lambda_0 < \lambda_1$ and there exist $x_1, x_2 \in \mathcal{H}_1$ and $y_1, y_2 \in \mathcal{H}_2$, $x_1, y_1, x_2, y_2 \neq 0$, with $\lambda_1 := \lambda_+(x_1, y_1)$ and $\lambda_2 := \lambda_+(x_2, y_2)$. Let $\big(x(t)\ y(t)\big)^{\mathrm{t}}$, $t \in [0,1]$, be a curve in $\mathcal{H}$ connecting $(x_1\ y_1)^{\mathrm{t}}$ and $(x_2\ y_2)^{\mathrm{t}}$ as in Lemma 1.10.1. Then $\lambda_+\big(x(t), y(t)\big)$, $t \in [0,1]$, is a curve in $W^2(\mathcal{A})$ connecting $\lambda_1$ and $\lambda_2$. This curve cannot stay in $\mathbb{R}$ since $\lambda_2 < \lambda_0 < \lambda_1$ and $\lambda_0 \notin \Lambda_+$; hence $\operatorname{dis}_{\mathcal{A}}\big(x(t), y(t)\big) < 0$ for some $t \in [0,1]$. Let $t_0 \in [0,1]$ be the minimal zero of $\operatorname{dis}_{\mathcal{A}}\big(x(\cdot), y(\cdot)\big)$. Then $\lambda_+\big(x(t_0), y(t_0)\big) = \lambda_-\big(x(t_0), y(t_0)\big) \in \Lambda_+ \cap \Lambda_-$. On the other hand, $\lambda_+\big(x(t), y(t)\big) \in \Lambda_+$, $t \in [0, t_0]$, and $\lambda_+\big(x(0), y(0)\big) = \lambda_1 > \lambda_0$; hence $\lambda_+\big(x(t_0), y(t_0)\big) > \lambda_0 \geq \sup \Lambda_-$, a contradiction to $\lambda_+\big(x(t_0), y(t_0)\big) \in \Lambda_-$. □

In the following we identify a subinterval $[\nu, \mu] \subset W^2(\mathcal{A}) \cap \mathbb{R}$ such that outside of this interval $\mathcal{A}$ possesses a local spectral function. The interval is defined such that its complement lies outside of the intersection of $\Lambda_-$ and $\Lambda_+$ and does not contain accumulation points of the non-real spectrum.

**Proposition 1.10.4** *Let $\mathcal{A}$ be $\mathcal{J}$-self-adjoint. Define*

$$\Lambda_0 := \big\{\lambda \in \mathbb{R} : \exists\,(\lambda_n)_1^\infty \subset W^2(\mathcal{A}) \setminus \mathbb{R},\ \lim_{n\to\infty} \lambda_n = \lambda\big\} \tag{1.10.1}$$

*and*

$$\begin{aligned} \nu &:= \min\big\{\inf \Lambda_+,\ \inf \Lambda_0,\ \max\{a_-, d_-\}\big\}, \\ \mu &:= \max\big\{\sup \Lambda_-,\ \sup \Lambda_0,\ \min\{a_+, d_+\}\big\}. \end{aligned} \tag{1.10.2}$$

*Then the interval $[\nu, \mu]$ satisfies the inclusions*

i) $\big[\max\{a_-, d_-\}, \min\{a_+, d_+\}\big] \subset [\nu, \mu] \subset \left[\dfrac{a_- + d_-}{2}, \dfrac{a_+ + d_+}{2}\right]$,

ii) $\Lambda_- \cap \Lambda_+ \ \subset\ [\inf \Lambda_+, \sup \Lambda_-]\ \subset\ [\nu, \mu]$.

***Proof.*** The left inclusion in i) and the right inclusion in ii) are immediate from the definition of $\nu$ and $\mu$. The left inclusion in ii) is obvious from the inequalities $\inf \Lambda_+ \geq \inf \Lambda_-$, $\sup \Lambda_+ \geq \sup \Lambda_-$. For the right inclusion in i), we observe that *e.g.* $\min\{a_+, d_+\} \leq (a_+ + d_+)/2$, $\lambda_-(x, y) \leq (a_+ + d_+)/2$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, by definition of $\lambda_-$, and $\operatorname{Re}\big(W^2(\mathcal{A}) \setminus \mathbb{R}\big) \subset \big[(a_- + d_-)/2, (a_+ + d_+)/2\big]$ by Proposition 1.3.9 ii). □

**Remark 1.10.5** The interval $[\nu, \mu]$ was defined differently in [LLMT05, (2.4)], not taking into account the accumulation of the non-real points of $W^2(\mathcal{A})$. It was assumed, but not proved there, that non-real points of $W^2(\mathcal{A})$ can accumulate at the real axis only within $\Lambda_- \cap \Lambda_+$. This question is still open and hence the definition of $\nu, \mu$ had to be modified here to ensure the completeness of the proof of [LLMT05, Theorem 3.1] (presented as Theorem 1.10.9 below).

In the following we introduce the notions of spectral points of definite type of $\mathcal{J}$-self-adjoint operators and of $\mathcal{J}$-nonnegative and $\mathcal{J}$-nonpositive subspaces (see [Lan82, Sections II.4, I.1]).

**Definition 1.10.6** Let $\mathcal{A} \in L(\mathcal{H})$ be $\mathcal{J}$-self-adjoint and let $[\cdot,\cdot] := (\mathcal{J}\cdot,\cdot)$.

i) An eigenvector $\mathbf{x_0}$ at an eigenvalue $\lambda_0$ of $\mathcal{A}$ is called of *$\mathcal{J}$-positive type* if

$$[\mathbf{x_0}, \mathbf{x_0}] > 0;$$

an eigenvalue $\lambda_0$ of $\mathcal{A}$ is called of *$\mathcal{J}$-positive type* if all corresponding eigenvectors are of $\mathcal{J}$-positive type.

iii) A spectral point $\lambda_0 \in \sigma_{\rm app}(\mathcal{A}) \cap \mathbb{R}$ is called of *$\mathcal{J}$-positive type* if

$$\|\mathbf{x_n}\| = 1,\ \lim_{n\to\infty} \|(\mathcal{A} - \lambda_0)\mathbf{x_n}\| = 0 \implies \liminf_{n\to\infty}\, [\mathbf{x_n}, \mathbf{x_n}] > 0.$$

The definition of eigenvectors, eigenvalues, and spectral points of *$\mathcal{J}$-negative type* is analogous.

**Definition 1.10.7** A subspace $\mathcal{L} \subset \mathcal{H}$ is called *$\mathcal{J}$-nonnegative* (*$\mathcal{J}$-positive, uniformly $\mathcal{J}$-positive*, respectively) if $[x,x] \geq 0$ for all $x \in \mathcal{L}$, $x \neq 0$ ($> 0$ or $\geq \gamma \|x\|^2$ with some $\gamma > 0$, respectively). A $\mathcal{J}$-nonnegative subspace $\mathcal{L}$ is called *maximal $\mathcal{J}$-nonnegative* if it is not properly contained in another $\mathcal{J}$-nonnegative subspace.

If for an interval $(\alpha, \beta) \subset \mathbb{R}$ the points of $(\alpha,\beta) \cap \sigma(\mathcal{A})$ are all of $\mathcal{J}$-positive or all of $\mathcal{J}$-negative type for $\mathcal{A}$, then $\mathcal{A}$ has a local spectral function $E_{\mathcal{A}}(\Delta)$ on $(\alpha,\beta)$ (see [LMM97, Theorem 3.1]); the spectral function is defined for all $\Delta$ belonging to the semi-ring $\mathcal{M}(\alpha,\beta)$ generated by all closed, open or semi-closed intervals whose closure is contained in $(\alpha,\beta)$.

The subspace $\mathcal{L}_\Delta(\mathcal{A}) := E_{\mathcal{A}}(\Delta)\mathcal{H}$ is called the *spectral subspace* of $\mathcal{A}$ corresponding to $\Delta$. If $\Delta \in \mathcal{M}(\alpha,\beta)$ is so that $\Delta \cap \sigma(\mathcal{A}) \neq \emptyset$ and consists of points of $\mathcal{J}$-positive type, then the corresponding spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ is a uniformly $\mathcal{J}$-positive subspace of $\mathcal{H}$ (see [Lan82, Remark 1]).

A useful property of $\mathcal{J}$-definite subspaces is that they admit angular operator representations (see *e.g.* [Lan82], [Bog74, Theorem II.11.7]).

**Remark 1.10.8** A closed subspace $\mathcal{L} \subset \mathcal{H}$ is $\mathcal{J}$-nonnegative if and only if there is a closed subspace $\mathcal{H}_1^{\mathcal{L}} \subset \mathcal{H}_1$ and a contraction $K : \mathcal{H}_1^{\mathcal{L}} \to \mathcal{H}_2$ with

$$\mathcal{L} = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1^{\mathcal{L}} \right\}.$$

The subspace $\mathcal{L}$ is $\mathcal{J}$-positive if and only if $K$ is a strict contraction (*i.e.* $\|Kx\| < \|x\|$ for all $x \in \mathcal{H}_1^{\mathcal{L}}$, $x \neq 0$), and $\mathcal{L}$ is uniformly $\mathcal{J}$-positive if and only if $K$ is a uniform contraction on $\mathcal{H}_1^{\mathcal{L}}$ (*i.e.* $\|K\| < 1$); the subspace $\mathcal{L}$ is maximal $\mathcal{J}$-nonnegative if and only if $\mathcal{H}_1^{\mathcal{L}} = \mathcal{H}_1$.

Now we are ready to classify spectral points of $\mathcal{J}$-self-adjoint block operator matrices according to their types.

**Theorem 1.10.9** *Let $\mathcal{A}$ be $\mathcal{J}$-self-adjoint and let $\nu$, $\mu$ be defined as in* (1.10.2). *Then*

i) *the spectral points of $\mathcal{A}$ in $(\mu, \infty)$ are of*

$$\mathcal{J}\text{-positive type} \quad \text{if} \;\; d_+ < a_+,$$
$$\mathcal{J}\text{-negative type} \quad \text{if} \;\; a_+ < d_+;$$

ii) *the spectral points of* $\mathcal{A}$ *in* $(-\infty,\nu)$ *are of*

$$\begin{aligned} &\mathcal{J}\text{-negative type} \quad \text{if } d_- < a_-, \\ &\mathcal{J}\text{-positive type} \quad \text{if } a_- < d_-; \end{aligned}$$

iii) $\mathcal{A}$ *has a local spectral function* $E_{\mathcal{A}}$ *on* $(-\infty,\nu)$ *and* $(\mu,\infty)$.

**Remark 1.10.10** From Proposition 1.10.3 i) it follows that

$$\begin{aligned} a_+ = d_+ &\implies \sigma(\mathcal{A}) \cap (\mu,\infty) = \emptyset, \\ a_- = d_- &\implies \sigma(\mathcal{A}) \cap (-\infty,\nu) = \emptyset. \end{aligned}$$

***Proof.*** i), ii) We restrict ourselves to the interval $(\mu,\infty)$ and to the case $d_+ < a_+$; all other cases are analogous.

Let $\lambda \in \sigma(\mathcal{A}) \cap (\mu,\infty)$. Then there exists a sequence $(\mathbf{x_n})_1^\infty \subset \mathcal{H}$, $\mathbf{x_n} = (x_n\ y_n)^{\mathrm{t}}$, such that $\|\mathbf{x_n}\|^2 = \|x_n\|^2 + \|y_n\|^2 = 1$ and

$$\mathbf{u_n} := (\mathcal{A}-\lambda)\mathbf{x_n} \longrightarrow 0, \quad n \to \infty. \tag{1.10.3}$$

Without loss of generality, we assume that $\lim_{n\to\infty}\|x_n\|$ exists. In order to prove that $\lambda$ is of $\mathcal{J}$-positive type, it suffices to show that $\lim_{n\to\infty}\|x_n\|^2 > 1/2$ since $[\mathbf{x_n},\mathbf{x_n}] = \|x_n\|^2 - \|y_n\|^2 = 2\|x_n\|^2 - 1$. Evidently, if $\lim_{n\to\infty}\|y_n\| = 0$, the claim is trivial. Otherwise, we let $\mathbf{u_n} = (u_n\ v_n)^{\mathrm{t}}$, take inner products of the rows in (1.10.3) with $x_n$ and $y_n$, respectively, and arrive at

$$(Ax_n,x_n) - \lambda\|x_n\|^2 + (By_n,x_n) \;=\; (u_n,x_n), \tag{1.10.4}$$

$$-(B^*x_n,y_n) + (Dy_n,y_n) - \lambda\|y_n\|^2 \;=\; (v_n,y_n). \tag{1.10.5}$$

If $\lim_{n\to\infty}\|x_n\|=0$, then (1.10.5) would imply $(Dy_n,y_n)/\|y_n\|^2 \to \lambda$, $n\to\infty$, a contradiction to $\lambda > d_+$. So, in the following, we may assume without loss of generality that $x_n, y_n \neq 0$ and $0 < \lim_{n\to\infty}\|x_n\| < 1$. If we set

$$a_n := \frac{(Ax_n,x_n)}{\|x_n\|^2}, \quad b_n := \frac{(By_n,x_n)}{\|x_n\|\,\|y_n\|}, \quad d_n := \frac{(Dy_n,y_n)}{\|y_n\|^2},$$

then (1.10.4), (1.10.5), and (1.10.3) imply that

$$\big(\mathcal{A}_{x_n,y_n}-\lambda\big)\begin{pmatrix}\|x_n\|\\ \|y_n\|\end{pmatrix} = \begin{pmatrix}a_n-\lambda & b_n\\ -\overline{b_n} & d_n-\lambda\end{pmatrix}\begin{pmatrix}\|x_n\|\\ \|y_n\|\end{pmatrix} = \begin{pmatrix}\frac{(u_n,x_n)}{\|x_n\|}\\ \frac{(v_n,y_n)}{\|y_n\|}\end{pmatrix} \longrightarrow 0,\ n\to\infty.$$

Therefore $\operatorname{dist}\big(\lambda, \sigma(\mathcal{A}_{x_n,y_n})\big) \to 0$, $n \to \infty$, by Lemma 1.3.2. Because $\lambda > \mu \geq \max\{\sup\Lambda_-, \sup\Lambda_0\}$, neither points of $\Lambda_-$ nor non-real points of $W^2(\mathcal{A})$ can accumulate at $\lambda$; hence $\lim_{n\to\infty}\lambda_+(x_n,y_n) = \lambda$.

Adding (1.10.4) and the complex conjugate of (1.10.5), we see that

$$a_n\|x_n\|^2 + d_n(1-\|x_n\|^2) - \lambda = (u_n, x_n) + (y_n, v_n) =: \varepsilon_n \longrightarrow 0, \quad n \to \infty.$$

Since $\lambda > \mu \geq d_+ \geq d_n$, there exists an $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$0 < \frac{\lambda - d_n + \varepsilon_n}{\|x_n\|^2} = a_n - d_n \leq a_+ - d_-, \quad \left|\lambda - \lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right| < \frac{\lambda - \mu}{4}. \tag{1.10.6}$$

Then we have, for all $n \geq n_0$,

$$\begin{aligned}\Big(\|x_n\|^2 - \frac{1}{2}\Big)\Big(a_+ - d_-\Big) \geq \Big(\|x_n\|^2 - \frac{1}{2}\Big)\Big(a_n - d_n\Big) &= \lambda - \frac{a_n + d_n}{2} + \varepsilon_n \\ &= \lambda - \frac{1}{2}\left(\lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \lambda_-\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right) + \varepsilon_n \\ &\geq \frac{\lambda - \mu}{2} - \frac{1}{2}\left(\lambda - \lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right) \\ &\geq \frac{\lambda - \mu}{4} > 0.\end{aligned}$$

Since $a_+ > d_+ \geq d_-$, we obtain the desired inequality $\lim_{n\to\infty} \|x_n\|^2 > 1/2$.

iii) The existence of the local spectral function on $(-\infty, \nu)$ and on $(\mu, \infty)$ follows from ii) (see [LMM97, Theorem 3.1 and Lemma 1.4]). □

**Corollary 1.10.11** *Let $\Delta$ be an interval with $\overline{\Delta} > \mu$ and let $\mathcal{L}_\Delta(\mathcal{A}) = E_{\mathcal{A}}(\Delta)\mathcal{H}$. If $d_+ < a_+$, then there exist a subspace $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$ and a strict contraction $K_1^\Delta \in L\big(\mathcal{H}_1^\Delta, \mathcal{H}_2\big)$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta\right\};$$

*if $a_+ < d_+$, then there exist a subspace $\mathcal{H}_2^\Delta \subset \mathcal{H}_2$ and a strict contraction $K_2^\Delta \in L\big(\mathcal{H}_2^\Delta, \mathcal{H}_1\big)$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{\begin{pmatrix} K_2^\Delta y \\ y \end{pmatrix} : y \in \mathcal{H}_2^\Delta\right\}.$$

***Proof.*** Theorem 1.10.9 shows that $\mathcal{L}_\Delta(\mathcal{A}) = E_{\mathcal{A}}(\Delta)\mathcal{H}$ is uniformly positive if $d_+ < a_+$ and $\Delta \cap \sigma(\mathcal{A}) \neq \emptyset$, and uniformly negative if $a_+ < d_+$ and $\Delta \cap \sigma(\mathcal{A}) \neq \emptyset$. Now both claims follow from Remark 1.10.8. □

As in the self-adjoint case (see Theorem 1.8.7), we call the subspaces $\mathcal{H}_1^\Delta$ and $\mathcal{H}_2^\Delta$ $\Delta$-spectral supporting subspaces of $\mathcal{A}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; for the following description in terms of the Schur complements, we restrict ourselves to the case $d_+ < a_+$.

**Theorem 1.10.12** *Suppose that $\mathcal{A}$ is $\mathcal{J}$-self-adjoint. Let $\Delta = [\alpha, \beta] > \mu$ and let $\Gamma_\Delta$ be a simply closed Jordan curve that surrounds $\Delta$, but no point of $\sigma(\mathcal{A}) \setminus \Delta$, and intersects $\mathbb{R}$ orthogonally in $\alpha$ and $\beta$. Define*

$$Q_\Delta := -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\Delta}' S_1(z)^{-1}\,\mathrm{d}z,$$

*where* $\int'$ *denotes the Cauchy principal value at* $\mathbb{R}$. *Then the range of* $Q_\Delta$ *is given by* $R\bigl(Q_\Delta\bigr) = \mathcal{H}_1^\Delta$.

***Proof.*** The proof follows the lines of the proof of Theorem 1.8.7 if we use the local spectral function $E_\mathcal{A}$ of $\mathcal{A}$ according to Theorem 1.10.9, introduce

$$\widehat{E}_\mathcal{A}(\Delta) := \frac{1}{2}\Bigl(E_\mathcal{A}(\Delta) + E_\mathcal{A}(\Delta^\circ)\Bigr) = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_\Delta}' (\mathcal{A}-z)^{-1}\,\mathrm{d}z$$

instead of $\widehat{P}_\Delta(\mathcal{A})$ and use the symmetries *e.g.* of $E_\mathcal{A}(\Delta)$ with respect to the indefinite inner product $[\cdot,\cdot]$ (see [LLMT05, Theorem 3.3]). □

The next theorem is the $\mathcal{J}$-self-adjoint analogue of Theorem 1.8.8.

**Theorem 1.10.13** *Let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint with* $d_+ < a_+$ *and let* $\Delta = [\alpha,\beta] > \mu$ *be an interval such that* $\alpha,\beta \in \rho(\mathcal{A})$. *Then*

$$\mathcal{H}_1 = \mathcal{L}_{(-\infty,0)}\bigl(S_1(\alpha)\bigr) \dotplus \mathcal{H}_1^\Delta \dotplus \mathcal{L}_{(0,\infty)}\bigl(S_1(\beta)\bigr).$$

***Proof.*** The proof is similar to the proof of Theorem 1.8.8 if we observe that, although the Schur complement need not be decreasing in the $\mathcal{J}$-self-adjoint case, the following weaker statement can be proved: If $\mu < \lambda_1 < \lambda_2$ and $\bigl(S_1(\lambda_2)x,x\bigr) \geq 0$ for some $x \in \mathcal{H}_1$, $x \neq 0$, then $\bigl(S_1(\lambda_1)x,x\bigr) > 0$. For more details we refer to [LLMT05, Section 4]. □

We conclude this section by investigating the corners of the zones $\Lambda_-$ and $\Lambda_+$ of the quadratic numerical range. If $a_- \neq d_-$ and $a_+ \neq d_+$, respectively, then the outer corners $\inf\Lambda_-$ and $\sup\Lambda_+$ are corners of $W^2(\mathcal{A})$ and hence, by Theorems 1.5.8 and 1.5.2, they belong to $\sigma(\mathcal{A})$, or even to $\sigma_\mathrm{p}(\mathcal{A})$ if $\inf\Lambda_- = \min\Lambda_-$ and $\sup\Lambda_+ = \max\Lambda_+$, respectively.

In the following theorem, we prove analogous statements for the interior corner $\sup\Lambda_-$ of $\Lambda_-$; the formulation of the analogue for $\inf\Lambda_+$ is obvious.

**Theorem 1.10.14** *Let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint. Suppose that* $\sup\Lambda_- > d_+$ *and that there exists a neighbourhood of* $\sup\Lambda_-$ *containing no non-real points of* $W^2(\mathcal{A})$. *Then* $\sup\Lambda_- \in \sigma(\mathcal{A})$.

*If, in addition,* $\sup\Lambda_- = \max\Lambda_-$, *then* $\sup\Lambda_- \in \sigma_\mathrm{p}(\mathcal{A})$. *In the latter case, if* $\sup\Lambda_- = \lambda_-(x_0,y_0)$ *for some* $x_0 \in \mathcal{H}_1$, $y_0 \in \mathcal{H}_2$, $x_0, y_0 \neq 0$, *then there is a* $\gamma \in \mathbb{C}$ *so that* $(x_0\ \ \gamma y_0)^\mathrm{t}$ *is an eigenvector of* $\mathcal{A}$ *corresponding to* $\sup\Lambda_-$.

***Proof.*** The proof is very similar to the proofs of Theorems 1.5.2 and 1.5.8; for details we refer the reader to the proof of [LLMT05, Theorem 2.7]. □

## 1.11 The block numerical range

The concept of quadratic numerical range for $2 \times 2$ block operator matrices has an obvious generalization to $n \times n$ block operator matrices. For this so-called block numerical range, we prove results on spectral inclusion, estimates of the resolvent, and inclusion theorems between block numerical ranges under refinements of the decomposition of the space (see [TW03]).

Let $n \in \mathbb{N}$, let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be complex Hilbert spaces, and consider $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. With respect to this decomposition, every operator $\mathcal{A} \in L(\mathcal{H})$ has an $n \times n$ block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \tag{1.11.1}$$

with entries $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$, $i, j = 1, \dots, n$. In the following we denote by $\mathcal{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n} := S_{\mathcal{H}_1} \times \cdots \times S_{\mathcal{H}_n} = \big\{ (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{H} : \|x_i\| = 1,\ i = 1, 2, \dots, n \big\}$ the product of the unit spheres $S_{\mathcal{H}_i}$ in $\mathcal{H}_i$; we also write$\mathcal{S}^n$ or $\mathcal{S}_{\mathcal{H}}$ instead of $\mathcal{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}$ if the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is clear (note the slight difference in notation between $\mathcal{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}$ and the unit sphere $S_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}$ in $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$).

**Definition 1.11.1** For $x = (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}$ we introduce the $n \times n$ matrix

$$\mathcal{A}_x := \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) \end{pmatrix} \in M_n(\mathbb{C}), \tag{1.11.2}$$

that is, $(\mathcal{A}_x)_{ij} := \big( A_{ij} x_j, x_i \big)$, $i, j = 1, \dots, n$. Then the set

$$W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}) := \bigcup_{x \in \mathcal{S}^n} \sigma_{\mathrm{p}}(\mathcal{A}_x) \tag{1.11.3}$$

is called *block numerical range* of $\mathcal{A}$ (with respect to the block operator matrix representation (1.11.1)). For a fixed decomposition of $\mathcal{H}$, we also write

$$W^n(\mathcal{A}) = W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}).$$

Clearly, since $\sigma_{\mathrm{p}}(\mathcal{A}_x) = \big\{ \lambda \in \mathbb{C} : \det(\mathcal{A}_x - \lambda) = 0 \big\}$ for all $x \in \mathcal{S}^n$, $W^n(\mathcal{A})$ has the equivalent representation

$$W^n(\mathcal{A}) = \big\{ \lambda \in \mathbb{C} : \exists\, x \in \mathcal{S}^n \ \det(\mathcal{A}_x - \lambda) = 0 \big\}. \tag{1.11.4}$$

**Remark 1.11.2** For $n = 1$ the block numerical range is just the usual numerical range, for $n = 2$ it is the quadratic numerical range introduced in Section 1.1. For $n = 3$, the block numerical range is also called *cubic numerical range* and for $n = 4$ *quartic numerical range*. If $\mathcal{A} \in M_n(\mathbb{C})$ is an $n \times n$ matrix, then $W^n(\mathcal{A})$ coincides with the set of eigenvalues of $\mathcal{A}$.

Like the numerical range and the quadratic numerical range, the block numerical range of a bounded block operator matrix $\mathcal{A}$ is bounded,

$$W^n(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \; |\lambda| \le \|\mathcal{A}\|\big\},$$

and closed if $\dim \mathcal{H} < \infty$. The former follows if we let $x = (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}^n$, $z = (z_1 \dots z_n)^{\mathrm{t}} \in \mathbb{C}^n$, $\|z\| = 1$, set $y_j := z_j x_j$, $j = 1, \dots, n$, $y := (y_1 \dots y_n)^{\mathrm{t}}$ so that $\|y\| = 1$, and observe that

$$\|\mathcal{A}_x z\|^2 = \sum_{i=1}^n \Big|\sum_{j=1}^n \big(A_{ij}x_j, x_i\big) z_j\Big|^2 \le \sum_{i=1}^n \Big|\Big|\sum_{j=1}^n A_{ij} y_j\Big|\Big|^2 ||x_i||^2 = \|\mathcal{A}y\|^2 \le \|\mathcal{A}\|^2.$$

If $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is decomposed into $n$ components, then the corresponding block numerical range consists of at most $n$ (connected) components; as in the case $n = 2$, this follows from the fact that the set of all matrices $\mathcal{A}_x$, $x \in \mathcal{S}^n$, is connected and from a continuity argument for the eigenvalues of matrices (see [Kat95, Theorem II.5.14] and [Wag00]). If, for example, $\mathcal{A}$ is upper or lower block triangular, then

$$W^n(\mathcal{A}) = W(A_{11}) \cup \cdots \cup W(A_{nn}).$$

This shows that, like the quadratic numerical range, $W^n(\mathcal{A})$ need not be convex; the next example shows that its components need not be so either.

**Example 1.11.3** Consider the $4 \times 4$ matrix

$$\mathcal{A}_7 = \left(\begin{array}{cc|c|c} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ \hline \mathrm{i} & \mathrm{i} & 1 & 0 \\ \hline \mathrm{i} & \mathrm{i} & 0 & -1 \end{array}\right)$$

with respect to $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$. The corresponding cubic numerical range has 3 components and none of them is convex (see Fig. 1.8).

**Proposition 1.11.4** *If $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, then*

i) $W^n(\mathcal{A}^*) = \big\{\overline{\lambda} \in \mathbb{C} : \lambda \in W^n(\mathcal{A})\big\} =: W^n(\mathcal{A})^*$.
ii) $\mathcal{A} = \mathcal{A}^* \implies W^n(\mathcal{A}) \subset \mathbb{R}$.

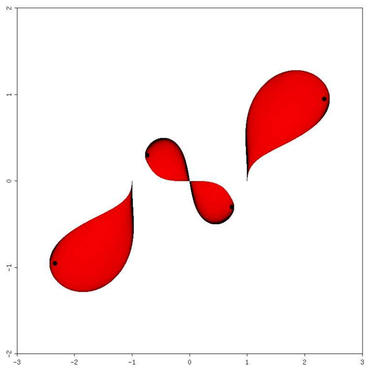

Figure 1.8 Cubic numerical range $W^3(\mathcal{A}_7) = W_{\mathbb{C}^2\oplus\mathbb{C}\oplus\mathbb{C}}(\mathcal{A}_7)$ of $\mathcal{A}_7$.

***Proof.*** For assertion i) we observe that $(\mathcal{A}_x)^* = (\mathcal{A}^*)_x$; assertion ii) is obvious since in this case all matrices $\mathcal{A}_x$ are symmetric. □

In Fig. 1.8 the eigenvalues of $\mathcal{A}_7$, which are marked by black dots, are obviously contained in $W^3(\mathcal{A}_7)$. In order to prove a general spectral inclusion theorem, we need the following generalization of Lemma 1.3.2.

**Lemma 1.11.5** *Let $\mathcal{M} \in M_n(\mathbb{C})$. If $\mathcal{M}$ is invertible, then*

$$\|\mathcal{M}^{-1}\| \leq \frac{\|\mathcal{M}\|^{n-1}}{|\det \mathcal{M}|}. \tag{1.11.5}$$

*For all $x \in \mathbb{C}^n$, $\|x\| = 1$, we have*

$$\operatorname{dist}\big(0, \sigma(\mathcal{M})\big) \leq \sqrt[n]{\|\mathcal{M}\|^{n-1}\|\mathcal{M}x\|}.$$

***Proof.*** The first estimate was proved in [Kat60, Lemma 1] (see also [Kat95, Section I.4.2, (4.12)] and note that $\mathbb{C}^n$ is a unitary space). The second statement is trivial if $\mathcal{M}$ is not invertible. If $\mathcal{M}$ is invertible and $x \in \mathbb{C}^n$, $\|x\| = 1$, then $\|\mathcal{M}x\| \geq \|\mathcal{M}^{-1}\|^{-1} > 0$. Denoting by $\lambda_1, \dots, \lambda_n$ the eigenvalues of $\mathcal{M}$ and using (1.11.5), we obtain

$$\begin{aligned}\big(\operatorname{dist}\big(0, \sigma(\mathcal{M})\big)\big)^n &= \Big(\min_{i=1}^{n} |\lambda_i|\Big)^n \leq |\lambda_1 \cdots \lambda_n| = |\det \mathcal{M}| \\ &\leq \frac{\|\mathcal{M}\|^{n-1}}{\|\mathcal{M}^{-1}\|} \leq \|\mathcal{M}\|^{n-1}\|\mathcal{M}x\|. \end{aligned}$$ □

The next theorem generalizes the spectral inclusion property of the numerical range ($n=1$, see (1.1.1)) and of the quadratic numerical range ($n=2$, see Theorem 1.3.1).

**Theorem 1.11.6** $\sigma_{\rm p}(\mathcal{A}) \subset W^n(\mathcal{A}),\ \sigma(\mathcal{A}) \subset \overline{W^n(\mathcal{A})}$.

***Proof.*** First let $\lambda \in \sigma_{\rm p}(\mathcal{A})$. Then there exists $x = (x_1 \dots x_n)^{\rm t} \in \mathcal{H}$, $x \neq 0$, such that $\mathcal{A}x - \lambda x = 0$. If we write $x_i = \|x_i\|\, \widehat{x}_i$ with $\widehat{x}_i \in \mathcal{H}_i$, $\|\widehat{x}_i\| = 1$, $i = 1, \dots, n$, then $\widehat{x} := (\widehat{x}_1 \dots \widehat{x}_n)^{\rm t} \in \mathcal{S}^n$, $(x_i, \widehat{x}_i) = \|x_i\|$, $i = 1, \dots, n$, and

$$\begin{aligned}
\big(\mathcal{A}_{\widehat{x}} - \lambda\big)\begin{pmatrix} \|x_1\| \\ \vdots \\ \|x_n\| \end{pmatrix} &= \begin{pmatrix} (A_{11}\widehat{x}_1, \widehat{x}_1) - \lambda & \cdots & (A_{1n}\widehat{x}_n, \widehat{x}_1) \\ \vdots & & \vdots \\ (A_{n1}\widehat{x}_1, \widehat{x}_n) & \cdots & (A_{nn}\widehat{x}_n, \widehat{x}_n) - \lambda \end{pmatrix}\begin{pmatrix} \|x_1\| \\ \vdots \\ \|x_n\| \end{pmatrix} \\
&= \begin{pmatrix} (A_{11}x_1, \widehat{x}_1) + \cdots + (A_{1n}x_n, \widehat{x}_1) - \lambda(x_1, \widehat{x}_1) \\ \vdots \\ (A_{n1}x_1, \widehat{x}_n) + \cdots + (A_{nn}x_n, \widehat{x}_n) - \lambda(x_n, \widehat{x}_n) \end{pmatrix} \\
&= \begin{pmatrix} \Big(\sum_{j=1}^n A_{1j}x_j - \lambda x_1, \widehat{x}_1\Big) \\ \vdots \\ \Big(\sum_{j=1}^n A_{nj}x_j - \lambda x_n, \widehat{x}_n\Big) \end{pmatrix} = \big(\mathcal{A}x - \lambda x, \widehat{x}\big) = 0.
\end{aligned}$$

Hence $\lambda \in \sigma_{\rm p}(\mathcal{A}_{\widehat{x}}) \subset W^n(\mathcal{A})$ by definition (1.11.3).

Now let $\lambda \in \sigma(\mathcal{A})$. Then we either have $\lambda \in \sigma_{\rm p}(\mathcal{A}^*)^*$ or $\lambda \in \sigma_{\rm app}(\mathcal{A})$ (the approximate point spectrum of $\mathcal{A}$, see (1.3.4)). If $\lambda \in \sigma_{\rm p}(\mathcal{A}^*)^*$, then the inclusion already proved and Proposition 1.11.4 i) yield $\overline{\lambda} \in \sigma_{\rm p}(\mathcal{A}^*) \subset W^n(\mathcal{A}^*) = W^n(\mathcal{A})^*$ and hence $\lambda \in W^n(\mathcal{A})$. If $\lambda \in \sigma_{\rm app}(\mathcal{A})$, then there is a sequence $\big(x^{(\nu)}\big)_1^\infty \subset \mathcal{H}$, $\|x^{(\nu)}\| = 1$, so that $\mathcal{A}x^{(\nu)} - \lambda x^{(\nu)} \to 0$, $\nu \to \infty$. If we write $x_i^{(\nu)} = \|x_i^{(\nu)}\|\, \widehat{x}_i^{(\nu)}$ with $\widehat{x}_i^{(\nu)} \in \mathcal{H}_i$, $\|\widehat{x}_i^{(\nu)}\| = 1$, $i = 1, \dots, n$, $\nu = 1, 2, \dots$, then $\widehat{x}^{(\nu)} := \big(\widehat{x}_1^{(\nu)} \dots \widehat{x}_n^{(\nu)}\big)^{\rm t} \in \mathcal{S}^n$ and, in a similar way as above, we obtain

$$\big(\mathcal{A}_{\widehat{x}^{(\nu)}} - \lambda\big)\begin{pmatrix} \|x_1^{(\nu)}\| \\ \vdots \\ \|x_n^{(\nu)}\| \end{pmatrix} = \Big(\mathcal{A}x^{(\nu)} - \lambda x^{(\nu)}, \widehat{x}^{(\nu)}\Big) \longrightarrow 0, \quad \nu \to \infty;$$

thus we have $\varepsilon_\nu := \Big\| \big(\mathcal{A}_{\widehat{x}^{(\nu)}} - \lambda\big)\big(\|x_1^{(\nu)}\| \dots \|x_n^{(\nu)}\|\big)^{\rm t} \Big\| \to 0$, $\nu \to \infty$. Since $\Big\| \big(\|x_1^{(\nu)}\| \dots \|x_n^{(\nu)}\|\big)^{\rm t} \Big\| = \big\|x^{(\nu)}\big\| = 1$, Lemma 1.11.5 implies that

$$\begin{aligned}
\operatorname{dist}\big(\lambda, \sigma\big(\mathcal{A}_{x^{(\nu)}}\big)\big) = \operatorname{dist}\big(0, \sigma\big(\mathcal{A}_{x^{(\nu)}} - \lambda\big)\big) &\leq \sqrt[n]{\|\mathcal{A}_{x^{(\nu)}} - \lambda\|^{n-1}\varepsilon_\nu} \\
&\leq \sqrt[n]{\big(\|\mathcal{A}\| + |\lambda|\big)^{n-1}\varepsilon_\nu} \longrightarrow 0, \quad \nu \to \infty,
\end{aligned}$$

and therefore

$$\lambda \in \overline{\bigcup_{\nu \in \mathbb{N}} \sigma(\mathcal{A}_{x^{(\nu)}})} \subset \overline{\bigcup_{x \in \mathcal{S}^n} \sigma(\mathcal{A}_x)} = \overline{W^n(\mathcal{A})}. \qquad \square$$

**Example 1.11.7** As an illustration of Theorem 1.11.6, we consider the block operator matrices

$$\mathcal{A}_8 = \left(\begin{array}{c|cc|c} 2 & \mathrm{i} & 1 & 0 \\ \hline \mathrm{i} & 2 & 0 & 1 \\ 1 & 0 & -2 & \mathrm{i} \\ \hline 0 & 1 & \mathrm{i} & -2 \end{array}\right), \quad \mathcal{A}_9 = \left(\begin{array}{c|c|cc} -2 & -1 & 1 & 0 \\ \hline -1 & -2 & 0 & 1 \\ \hline -2 & -1 & 0 & -3\mathrm{i} \\ -1 & -2 & 3\mathrm{i} & 0 \end{array}\right).$$

Figure 1.9 shows their eigenvalues marked by black dots and their cubic numerical ranges. Note that the horizontal line in the right picture is part of the cubic numerical range.

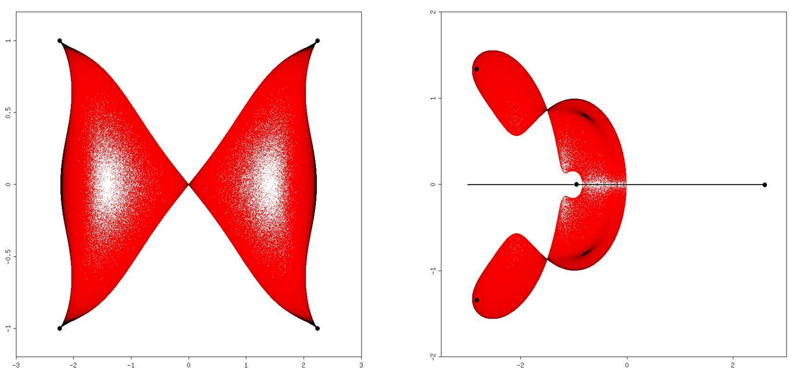

Figure 1.9 Cubic numerical ranges $W_{\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{C}}(\mathcal{A}_8)$, $W_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^2}(\mathcal{A}_9)$, and eigenvalues.

The next theorem generalizes Theorem 1.1.9; it shows that the block numerical range of a principal minor of an $n \times n$ block operator matrix $\mathcal{A}$ is contained in $W^n(\mathcal{A})$ if a certain dimension condition holds.

**Theorem 1.11.8** *Let $k \in \mathbb{N}$, $1 \le k \le n$, $1 \le i_1 < \cdots < i_k \le n$, and denote by $P : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \to \mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_k}$ the projection onto the components $i_1, \ldots, i_k$ of $\mathcal{H}$.*
*If there exists an enumeration $i'_1, \ldots, i'_{n-k}$ of the elements of the set $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ with $\dim \mathcal{H}_{i'_j} \ge n-(j-1)$, $j = 1, \ldots, n-k$, then*

$$W_{\mathcal{H}_{i_1}\oplus\cdots\oplus\mathcal{H}_{i_k}}(P\mathcal{A}P) \subset W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}).$$

***Proof.*** For $k = n$, the statement is trivial. For $k = n-1$ there is an $i \in \{1, \ldots, n\}$ such that $\{i_1, \ldots, i_k\} \cup \{i\} = \{1, \ldots, n\}$. If we denote $\mathcal{H}'_i := \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{i-1} \oplus \mathcal{H}_{i+1} \oplus \cdots \oplus \mathcal{H}_n$ and $\mathcal{A}'_i := P\mathcal{A}P$, then

$$\mathcal{A}_i' = \begin{pmatrix} A_{11} & \cdots & A_{1,i-1} & A_{1,i+1} & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{i-1,1} & \cdots & A_{i-1,i-1} & A_{i-1,i+1} & \cdots & A_{i-1,n} \\ A_{i+1,1} & \cdots & A_{i+1,i-1} & A_{i+1,i+1} & \cdots & A_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,i-1} & A_{n,i+1} & \cdots & A_{nn} \end{pmatrix}.$$

Now let $\lambda \in W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_{i-1}\oplus\mathcal{H}_{i+1}\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}_i')$. Then there exists an element $x' = (x_1 \dots x_{i-1}\, x_{i+1} \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}_i'}$ with $\det\big((\mathcal{A}_i')_{x'} - \lambda\big) = 0$. Since

$$\dim \operatorname{span}\big\{A_{i1}x_1, \dots, A_{i,i-1}x_{i-1}, A_{i,i+1}x_{i+1}, \dots, A_{in}x_n\big\} \le n-1 < \dim \mathcal{H}_i$$

by assumption, there is an $x_i \in \mathcal{H}_i$, $\|x_i\| = 1$, with $(\mathcal{A}_x)_{ij} = (A_{ij}x_j, x_i) = 0$ for $j = 1, \dots, i-1, i+1, \dots, n$. Then we have $x := (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}}$ and

$$\mathcal{A}_x = \begin{pmatrix} (\mathcal{A}_x)_{11} & \cdots & (\mathcal{A}_x)_{1,i-1} & (\mathcal{A}_x)_{1i} & (\mathcal{A}_x)_{1,i+1} & \cdots & (\mathcal{A}_x)_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \\ (\mathcal{A}_x)_{i-1,1} & \cdots & (\mathcal{A}_x)_{i-1,i-1} & (\mathcal{A}_x)_{i-1,i} & (\mathcal{A}_x)_{i-1,i+1} & \cdots & (\mathcal{A}_x)_{i-1,n} \\ 0 & \cdots & 0 & (\mathcal{A}_x)_{ii} & 0 & \cdots & 0 \\ (\mathcal{A}_x)_{i+1,1} & \cdots & (\mathcal{A}_x)_{i+1,i-1} & (\mathcal{A}_x)_{i+1,i} & (\mathcal{A}_x)_{i+1,i+1} & \cdots & (\mathcal{A}_x)_{i+1,n} \\ \vdots & & \vdots & \vdots & \dots & & \\ (\mathcal{A}_x)_{n1} & \cdots & (\mathcal{A}_x)_{n,i-1} & (\mathcal{A}_x)_{ni} & (\mathcal{A}_x)_{n,i+1} & \cdots & (\mathcal{A}_x)_{nn} \end{pmatrix}.$$

Thus $\det(\mathcal{A}_x - \lambda) = \big((\mathcal{A}_x)_{ii} - \lambda\big) \det\big((\mathcal{A}_i')_{x'} - \lambda\big) = 0$ and, consequently, $\lambda \in W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A})$. The case $k < n-1$ follows by induction. □

The particular case $k = 1$ of Theorem 1.11.8 shows that, under a certain dimension condition, the numerical ranges of the diagonal entries $A_{ii}$ of $\mathcal{A}$ are contained in the block numerical range of $\mathcal{A}$ (compare Theorem 1.1.9 and Corollary 1.1.10 for the case $n = 2$).

**Corollary 1.11.9** *Let $i_0 \in \mathbb{N}$. If there exists an enumeration $i_1', \dots, i_{n-1}'$ of $\{1, \dots, i_0 - 1, i_0 + 1, \dots, n\}$ with $\dim \mathcal{H}_{i_j'} \ge n - (j-1)$, $j = 1, \dots, n-1$, then*

$$W(A_{i_0 i_0}) \subset W^n(\mathcal{A});$$

*in particular, if $\dim \mathcal{H}_i \ge n$ for $i = 1, \dots, n$, then*

$$W(A_{ii}) \subset W^n(\mathcal{A}), \quad i = 1, \dots, n.$$

**Corollary 1.11.10** *Suppose that $\dim \mathcal{H}_i \ge n$, $i = 1, \dots, n$, and that $W^n(\mathcal{A}) = \mathcal{F}_1 \,\dot\cup \cdots \dot\cup\, \mathcal{F}_n$ consists of $n$ disjoint components. Then there exists a permutation $\pi$ of $\{1, \dots, n\}$ such that $W(A_{ii}) \subset \mathcal{F}_{\pi(i)}$, $i = 1, \dots, n$.*

***Proof.*** Let $x_1 \in S_{\mathcal{H}_1}$ be arbitrary. Then, by the dimension condition, we can recursively choose $x_k \in S_{\mathcal{H}_k}$, $k = 2, \dots, n$, in such a way that $x_k \perp \{A_{k1}x_1, \dots, A_{k,k-1}x_{k-1}\}$. Since $W^n(\mathcal{A}) = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_n$, every matrix $\mathcal{A}_x$, $x \in \mathcal{S}_{\mathcal{H}}$, has exactly one eigenvalue in each component of $W^n(\mathcal{A})$. In particular, if we let $x := (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}}$, then

$$\mathcal{A}_x = \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (A_{nn}x_n, x_n) \end{pmatrix};$$

hence there exists a permutation $\pi$ of $\{1, \dots, n\}$ with $(A_{ii}x_i, x_i) \in \mathcal{F}_{\pi(i)}$ for $i = 1, \dots, n$. By Corollary 1.11.9 we have $W(A_{ii}) \subset W^n(\mathcal{A})$ for $i = 1, \dots, n$; since $W(A_{ii})$ is convex, the assertion follows. □

The dimension condition in Theorem 1.11.8 cannot be dropped; this can be seen from the following example.

**Example 1.11.11** We reconsider the matrix $\mathcal{A}_5$ from Example 1.5.5, now with the $3 \times 3$ block decomposition in $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$, and its principal minor $\mathcal{A}_5'$ given by

$$\mathcal{A}_5 = \left(\begin{array}{c|cc|c} 1 & 3+\mathrm{i} & 2 & \mathrm{i} \\ \hline 3+\mathrm{i} & 1 & \mathrm{i} & 2 \\ -2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \hline \mathrm{i} & -2 & 3+\mathrm{i} & 1 \end{array}\right), \quad \mathcal{A}_5' = \left(\begin{array}{cc|c} 1 & \mathrm{i} & 2 \\ \mathrm{i} & 1 & 3+\mathrm{i} \\ \hline -2 & 3+\mathrm{i} & 1 \end{array}\right).$$

Figure 1.10 shows that the quadratic numerical range of $\mathcal{A}_5'$ is not contained in the cubic numerical range of $\mathcal{A}_5$; here $n=3$, $k=2$, $i_1=2$, $i_2=3$, $i_1'=1$ and so the dimension condition $\dim \mathcal{H}_1 \geq 3$ of Theorem 1.11.8 is violated.

Next we consider the behaviour of the block numerical range under refinements of the decomposition $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ of $\mathcal{H}$. Theorem 1.11.13 below is a generalization of the fact that the quadratic numerical range is contained in the numerical range (see Theorem 1.1.8).

**Definition 1.11.12** Let $n$, $\widetilde{n} \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n = \widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ with Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$ and $\widetilde{\mathcal{H}}_1, \dots, \widetilde{\mathcal{H}}_{\widetilde{n}}$. Then $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ is called a *refinement* of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ if $n \leq \widetilde{n}$ and there exist integers $0 = i_0 < \cdots < i_n = \widetilde{n}$ with $\mathcal{H}_k = \widetilde{\mathcal{H}}_{i_{k-1}+1} \oplus \cdots \oplus \widetilde{\mathcal{H}}_{i_k}$ for all $k = 1, \dots, n$.

**Theorem 1.11.13** *If $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ is a refinement of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, then*

$$W_{\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}}(\mathcal{A}) \subset W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}),$$

*or, briefly,*

$$W^{\widetilde{n}}(\mathcal{A}) \subset W^n(\mathcal{A}), \quad \widetilde{n} \geq n.$$

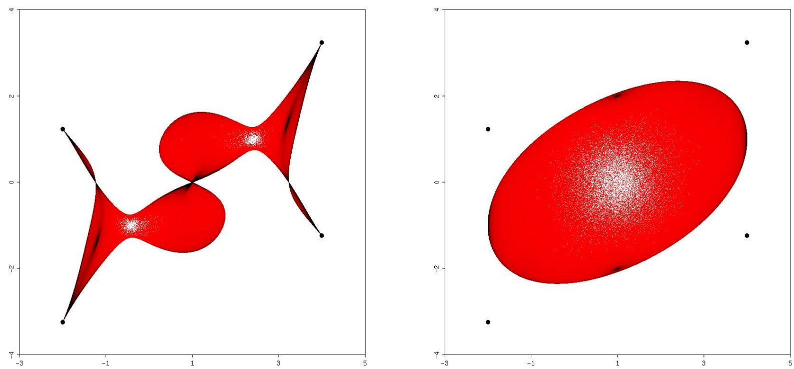


Figure 1.10 $W_{\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{C}}(\mathcal{A}_5)$ and $W_{\mathbb{C}^2\oplus\mathbb{C}}(\mathcal{A}_5')$

***Proof.*** It is sufficient to consider the case $\widetilde{n} = n+1$; the general case easily follows by induction. If $\widetilde{n} = n+1$, there exists a $k \in \{1,\dots,n\}$ such that the refinement $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ of $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is of the form $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k-1} \oplus \mathcal{H}_k^1 \oplus \mathcal{H}_k^2 \oplus \mathcal{H}_{k+1} \oplus \cdots \oplus \mathcal{H}_n$ where $\mathcal{H}_k = \mathcal{H}_k^1 \oplus \mathcal{H}_k^2$. With respect to this refined decomposition, $\mathcal{A}$ has the representation

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1k}^1 & A_{1k}^2 & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{k1}^1 & \cdots & A_{kk}^{11} & A_{kk}^{12} & \cdots & A_{kn}^1 \\ A_{k1}^2 & \cdots & A_{kk}^{21} & A_{kk}^{22} & \cdots & A_{kn}^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nk}^1 & A_{nk}^2 & \cdots & A_{nn} \end{pmatrix}$$

with $A_{kk}^{st} \in L(\mathcal{H}_k^t, \mathcal{H}_k^s)$, $A_{ki}^t \in L(\mathcal{H}_i, \mathcal{H}_k^t)$, $A_{jk}^s \in L(\mathcal{H}_k^s, \mathcal{H}_j)$, $k,i,j = 1,\dots,n$, $s,t = 1,2$. For the entries $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$, $i,j = 1,\dots,n$, of the representation (1.11.1) of $\mathcal{A}$ with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, we have

$$A_{kk} = \begin{pmatrix} A_{kk}^{11} & A_{kk}^{12} \\ A_{kk}^{21} & A_{kk}^{22} \end{pmatrix}, \quad A_{ki} = \begin{pmatrix} A_{ki}^1 \\ A_{ki}^2 \end{pmatrix}, \quad A_{jk} = \begin{pmatrix} A_{jk}^1 & A_{jk}^2 \end{pmatrix}.$$

By Theorem 1.11.6 about the spectral inclusion, we conclude that

$$\begin{aligned} & W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}) \\ &= \bigcup \left\{ \sigma(\mathcal{A}_x) : x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n} \right\} \\ &\subset \bigcup \left\{ W_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}(\mathcal{A}_x) : x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n} \right\}. \end{aligned}$$

The theorem is proved if we show that, for $x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n}$,

$$W_{\mathbb{C}\times\cdots\times\mathbb{C}^2\times\cdots\times\mathbb{C}}(\mathcal{A}_x) \subset \bigcup \left\{\sigma(\mathcal{A}_y) : y \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}\right\} = W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}).$$

To this end, let $x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n}$, $x = (x_1 \dots x_k^1\, x_k^2 \dots x_n)^{\mathrm{t}} \in \mathcal{H}$ with $\|x_1\| = \cdots = \|x_k^1\| = \|x_k^2\| = \cdots = \|x_n\| = 1$. Then

$$\mathcal{A}_x = \left(\begin{array}{c|c|cc|c|c}
(A_{11}x_1, x_1) & \cdots & (A_{1k}^1 x_k^1, x_1) & (A_{1k}^2 x_k^2, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \hline
\vdots & & \vdots & \vdots & & \vdots \\ \hline
(A_{k1}^1 x_1, x_k^1) & \cdots & (A_{kk}^{11} x_k^1, x_k^1) & (A_{kk}^{12} x_k^2, x_k^1) & \cdots & (A_{kn}^1 x_n, x_k^1) \\
(A_{k1}^2 x_1, x_k^2) & \cdots & (A_{kk}^{21} x_k^1, x_k^2) & (A_{kk}^{22} x_k^2, x_k^2) & \cdots & (A_{kn}^2 x_n, x_k^2) \\ \hline
\vdots & & \vdots & \vdots & & \vdots \\ \hline
(A_{n1}x_1, x_n) & \cdots & (A_{nk}^1 x_k^1, x_n) & (A_{nk}^2 x_k^2, x_n) & \cdots & (A_{nn}x_n, x_n)
\end{array}\right)$$

$$=: \begin{pmatrix} B_{11} & \cdots & B_{1k} & \cdots & B_{1n} \\ \vdots & & \vdots & & \vdots \\ B_{k1} & \cdots & B_{kk} & \cdots & B_{kn} \\ \vdots & & \vdots & & \vdots \\ B_{n1} & \cdots & B_{nk} & \cdots & B_{nn} \end{pmatrix} =: \mathcal{B} \in L\big(\mathbb{C} \oplus \cdots \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}\big).$$

Now let $z \in \mathcal{S}_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}$ be arbitrary. If we find a $y \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}$ with $\mathcal{B}_z = \mathcal{A}_y$, then $\sigma\big((\mathcal{A}_x)_z\big) = \sigma(\mathcal{B}_z) = \sigma(\mathcal{A}_y)$ and hence

$$\begin{aligned} W_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}(\mathcal{A}_x) &= \bigcup \left\{\sigma\big((\mathcal{A}_x)_z\big) : z \in \mathcal{S}_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}\right\} \\ &\subset \bigcup \left\{\sigma(\mathcal{A}_y) : y \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}\right\}, \end{aligned}$$

as required. To this end, let $z = (z_1 \dots z_k \dots z_n)^{\mathrm{t}} \in \mathbb{C} \oplus \cdots \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}$, $z_k = \big(z_k^1\ z_k^2\big)^{\mathrm{t}} \in \mathbb{C}^2$, with $|z_1|^2 = \cdots = \|z_k\|^2 = \cdots = |z_n|^2 = 1$. Then

$$\mathcal{B}_z = \begin{pmatrix} (B_{11}z_1, z_1) & \cdots & (B_{1k}z_k, z_1) & \cdots & (B_{1n}z_n, z_1) \\ \vdots & & \vdots & & \vdots \\ (B_{k1}z_1, z_k) & \cdots & (B_{kk}z_k, z_k) & \cdots & (B_{kn}z_n, z_k) \\ \vdots & & \vdots & & \vdots \\ (B_{n1}z_1, z_n) & \cdots & (B_{nk}z_k, z_n) & \cdots & (B_{nn}z_n, z_n) \end{pmatrix}.$$

Set

$$y_i := z_i x_i, \quad i = 1, \dots, n,\ i \neq k, \quad y_k := (y_k^1\ y_k^2)^{\mathrm{t}} := \big(z_k^1 x_k^1\ z_k^2 x_k^2\big).$$

Then it is not difficult to check that $\|y_i\| = 1$, $i = 1, \dots, n$, and hence $y := (y_1 \dots y_k \dots y_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}$. With this choice of $y$, we obtain the desired equality $\mathcal{B}_z = \mathcal{A}_y$. For example, for $k = 1, \dots, n$, the $k$-th diagonal elements $(B_z)_{kk}$ of $B_z$ and $(\mathcal{A}_y)_{kk}$ of $\mathcal{A}_y$ coincide since

$$\begin{aligned}
(B_z)_{kk} &= \left( \begin{pmatrix} (A_{kk}^{11} x_k^1, x_k^1) z_k^1 + (A_{kk}^{12} x_k^2, x_k^1) z_k^2 \\ (A_{kk}^{21} x_k^1, x_k^2) z_k^1 + (A_{kk}^{22} x_k^2, x_k^2) z_k^2 \end{pmatrix}, \begin{pmatrix} z_k^1 \\ z_k^2 \end{pmatrix} \right) \\
&= \big( (A_{kk}^{11} y_k^1, x_k^1) + (A_{kk}^{12} y_k^2, x_k^1) \big) \overline{z_k^1} + \big( (A_{kk}^{21} y_k^1, x_k^2) + (A_{kk}^{22} y_k^2, x_k^2) \big) \overline{z_k^2} \\
&= \big( A_{kk}^{11} y_k^1 + A_{kk}^{12} y_k^2, y_k^1 \big) + \big( A_{kk}^{21} y_k^1 + A_{kk}^{22} y_k^2, y_k^2 \big) \\
&= \left( \begin{pmatrix} A_{kk}^{11} y_k^1 + A_{kk}^{12} y_k^2 \\ A_{kk}^{21} y_k^1 + A_{kk}^{22} y_k^2 \end{pmatrix}, \begin{pmatrix} y_k^1 \\ y_k^2 \end{pmatrix} \right) = (A_{kk} y_k, y_k) = (\mathcal{A}_y)_{kk};
\end{aligned}$$

the proof for the other entries is similar. □

**Example 1.11.14** As an illustration for Theorem 1.11.13, we reconsider the matrix $\mathcal{A}_3$ from Example 1.3.3:

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & \mathrm{i} & 5\mathrm{i} \\ -1 & -2 & -5\mathrm{i} & \mathrm{i} \end{pmatrix}. \tag{1.11.6}$$

Its block numerical ranges with respect to the four successively refined decompositions $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ (the first one being the numerical range and the last one the spectrum) are displayed in Fig. 1.11 below.

**Remark 1.11.15** In [FH08], K.-H. Förster and N. Hartanto considered the block numerical range of (entrywise) nonnegative matrices. They developed a Perron-Frobenius theory for it, thus generalizing corresponding results for the spectrum and the numerical range.

The estimate for the resolvent of a block operator matrix in terms of the quadratic numerical range also generalizes to the block numerical range. For the proof we need the following generalization of Lemma 1.4.2.

**Lemma 1.11.16** *Let $\mathcal{A}_{(\cdot)} : \mathcal{S}^n \to M_n(\mathbb{C})$ be uniformly bounded from below, i.e. assume there exists a $\delta > 0$ such that for all $x \in \mathcal{S}^n$*

$$\|\mathcal{A}_x \alpha\| \geq \delta \, \|\alpha\|, \quad \alpha \in \mathbb{C}^n. \tag{1.11.7}$$

*Then*

$$\|\mathcal{A} y\| \geq \delta \, \|y\|, \quad y \in \mathcal{H};$$

*if, in addition, $\mathcal{A}$ is boundedly invertible, then $\|\mathcal{A}^{-1}\| \leq \delta^{-1}$.*

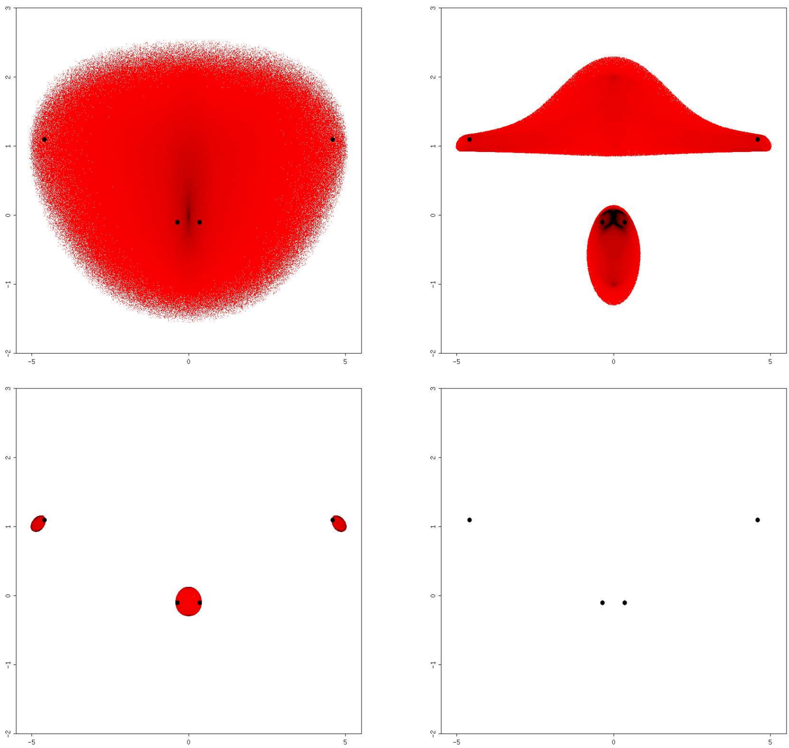

Figure 1.11 $W_{\mathbb{C}^4}(\mathcal{A}_3)$, $W_{\mathbb{C}^2\oplus\mathbb{C}^2}(\mathcal{A}_3)$, $W_{\mathbb{C}^2\oplus\mathbb{C}\oplus\mathbb{C}}(\mathcal{A}_3)$, and $W_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}}(\mathcal{A}_3)$.

***Proof.*** Let $y = (y_1 \dots y_n)^{\mathrm{t}} \in \mathcal{H}$ be arbitrary, write $y_i = \|y_i\|\,\widehat{y}_i$ with $\widehat{y}_i \in \mathcal{H}_i$, $\|\widehat{y}_i\| = 1$, $i = 1,\dots,n$, and set $\alpha := \big(\|y_1\| \dots \|y_n\|\big)^{\mathrm{t}} \in \mathbb{C}^n$. Then $\widehat{y} := (\widehat{y}_1 \dots \widehat{y}_n)^{\mathrm{t}} \in \mathcal{S}^n$ and hence, by assumption (1.11.7), we have $\|\mathcal{A}_{\widehat{y}}\,\alpha\|^2 \geq \delta^2\|\alpha\|^2 = \delta^2\|y\|^2$. Together with the equalities

$$\begin{aligned}
\|\mathcal{A}_{\widehat{y}}\,\alpha\|^2 &= \left\| \begin{pmatrix} (A_{11}\widehat{y}_1, \widehat{y}_1)\|y_1\| + \cdots + (A_{1n}\widehat{y}_n, \widehat{y}_1)\|y_n\| \\ \vdots \\ (A_{n1}\widehat{y}_1, \widehat{y}_n)\|y_1\| + \cdots + (A_{nn}\widehat{y}_n, \widehat{y}_n)\|y_n\| \end{pmatrix} \right\|^2 \\
&= \sum_{i=1}^n \left| \Big( \sum_{j=1}^n A_{ij} y_j, \widehat{y}_i \Big) \right|^2 \leq \sum_{i=1}^n \left\| \sum_{j=1}^n A_{ij} y_j \right\|^2 \|\widehat{y}_i\|^2 \\
&= \sum_{i=1}^n \left\| \sum_{j=1}^n A_{ij} y_j \right\|^2 = \|\mathcal{A} y\|^2,
\end{aligned}$$

the desired estimate follows. The last claim is obvious. □

The following theorem generalizes the resolvent estimate in terms of the numerical range ($n=1$, see (1.1.2)) and in terms of the quadratic numerical range ($n=2$, see Theorem 1.4.1).

**Theorem 1.11.17** *The resolvent of $\mathcal{A}$ admits the estimate*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\big(\|\mathcal{A}\|+|\lambda|\big)^{n-1}}{\operatorname{dist}\big(\lambda, W^n(\mathcal{A})\big)^n}, \qquad \lambda \notin \overline{W^n(\mathcal{A})}. \tag{1.11.8}$$

*More exactly, if $\mathcal{F}_1,\dots,\mathcal{F}_s$ are the components of $\overline{W^n(\mathcal{A})}$, then there are integers $n_j$, $j=1,\dots,s$, with $\sum_{j=1}^s n_j = n$ such that*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\|\mathcal{A}-\lambda\|^{n-1}}{\prod_{j=1}^s \operatorname{dist}\big(\lambda,\mathcal{F}_j\big)^{n_j}}, \quad \lambda \notin \overline{W^n(\mathcal{A})}; \tag{1.11.9}$$

*in particular, if $W^n(\mathcal{A})$ consists of $n$ components, then*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\|\mathcal{A}-\lambda\|^{n-1}}{\prod_{j=1}^n \operatorname{dist}(\lambda,\mathcal{F}_j)}, \qquad \lambda \notin \overline{W^n(\mathcal{A})}.$$

***Proof.*** Let $\lambda \notin \overline{W^n(\mathcal{A})}$. If $\mathcal{F}_1,\dots,\mathcal{F}_s$ are the components of $\overline{W^n(\mathcal{A})}$, then there are integers $n_j$, $j=1,\dots,s$, with $\sum_{j=1}^s n_j = n$ such that each matrix $\mathcal{A}_x$, $x\in\mathcal{S}^n$, has exactly $n_j$ eigenvalues in $\mathcal{F}_j$ for all $j=1,\dots,s$. Now let $x\in\mathcal{S}^n$ and let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of $\mathcal{A}_x$. Then there exists a partition $I_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} I_s = \{1,\dots,n\}$ so that $\lambda_i\in\mathcal{F}_j$ if and only if $i\in I_j$. Then $n_j = \# I_j$, $j=1,\dots,s$, and

$$\big|\det(\mathcal{A}_x-\lambda)\big| = |\lambda-\lambda_1|\cdots|\lambda-\lambda_n| = \prod_{j=1}^s\prod_{i\in I_j}|\lambda-\lambda_i| \ge \prod_{j=1}^s \operatorname{dist}(\lambda,\mathcal{F}_j)^{n_j} > 0$$

for $x\in\mathcal{S}^n$ since $\lambda\notin\overline{W^n(\mathcal{A})}$; in particular, $\mathcal{A}_x-\lambda$ is invertible. This and Lemma 1.11.5 now imply that

$$\big\|(\mathcal{A}-\lambda)_x^{-1}\big\| \le \frac{\|(\mathcal{A}-\lambda)_x\|^{n-1}}{|\det(\mathcal{A}_x-\lambda)|} \le \frac{\|\mathcal{A}-\lambda\|^{n-1}}{\prod_{j=1}^s \operatorname{dist}(\lambda,\mathcal{F}_j)^{n_j}} \tag{1.11.10}$$

for all $x\in\mathcal{S}^n$. Since $\lambda\notin\overline{W^n(\mathcal{A})}$, we have $\lambda\in\rho(A)$ by Theorem 1.11.6 and hence $\mathcal{A}-\lambda$ is invertible. Using this and (1.11.10), we obtain the second assertion of the theorem from Lemma 1.11.16. The first and the third estimate are immediate consequences of the second inequality. □

As for the numerical range and the quadratic numerical range, the estimate of the resolvent yields an upper bound for the length of Jordan chains at boundary points of the block numerical range (compare Corollary 1.4.8).

**Proposition 1.11.18** *Let $\lambda_0\in\sigma_{\mathrm{p}}(\mathcal{A})$. If $\lambda_0\in\partial W^n(\mathcal{A})$ has the exterior cone property, then the length of a Jordan chain at $\lambda_0$ is at most $n$.*

*More exactly, if* $\overline{W^n(\mathcal{A})} = \mathcal{F}_1 \dot\cup \dots \dot\cup \mathcal{F}_s$ *consists of* $s$ *disjoint components, such that* $\lambda_0 \in \mathcal{F}_{j_0}$*, and the integers* $n_j$*,* $j = 1, \dots, s$*, are as in the proof of* Theorem 1.11.17*, then the length of a Jordan chain at* $\lambda_0$ *is at most* $n_{j_0}$*. In particular, if* $\overline{W^n(\mathcal{A})}$ *consists of* $n$ *components, then the length of a Jordan chain at* $\lambda_0$ *is at most one, i.e. there are no associated vectors at* $\lambda_0$*.*

***Proof.*** The proof is completely analogous to the proof of Corollary 1.4.8 for the quadratic numerical range (see [TW03, Proposition 4.4]). □

**Remark 1.11.19** The block diagonalization theorem (see Theorem 1.7.1 and Corollary 1.7.2) was generalized recently to the $n \times n$ case by M. Wagenhofer (see the PhD thesis [Wag07]). He did not only assume that $W^n(\mathcal{A})$ has $n$ disjoint components, but that they are separated in some stronger sense.

## 1.12 Numerical ranges of operator polynomials

A special class of $n \times n$ block operator matrices, so-called *companion operators*, arises as linearizations of operator polynomials of degree $n$. Here we study the relation between the block numerical range of a companion operator and the numerical range of the corresponding operator polynomial.

Let $\mathcal{H}_0$ be a complex Hilbert space, let $A_i \in L(\mathcal{H}_0)$, $i = 0, \dots, n-1$, and set $A := (A_0, \dots, A_{n-1})$. Consider the operator polynomial $P_A$ given by

$$P_A(\lambda) := \lambda^n I + \lambda^{n-1} A_{n-1} + \dots + \lambda A_1 + A_0, \quad \lambda \in \mathbb{C}.$$

The companion operator $\mathcal{C}^A$ of $P_A$ is the $n \times n$ block operator matrix in the Hilbert space $\mathcal{H} = \mathcal{H}_0^n = \mathcal{H}_0 \oplus \dots \oplus \mathcal{H}_0$ given by

$$\mathcal{C}^A := \begin{pmatrix} 0 & I & \cdots & \cdots & 0 \\ \vdots & 0 & I & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & I \\ -A_0 & -A_1 & \cdots & -A_{n-2} & -A_{n-1} \end{pmatrix}.$$

It is well-known that the spectral properties of $P_A$ and its companion operator $\mathcal{C}^A$ are intimately related (see [Mül56], [Mar88, § 12.1]); in particular, $\sigma(P_A) = \sigma(\mathcal{C}^A)$ and $\sigma_p(P_A) = \sigma_p(\mathcal{C}^A)$.

The numerical range of the operator polynomial $P_A$ is given by $W(P_A) := \{\lambda \in \mathbb{C} : \exists\, f \in \mathcal{H},\, f \neq 0,\, (P_A(\lambda)f, f) = 0\}$ (see (1.6.1)). It is not difficult to check that $W(P_A) \subset W(\mathcal{C}^A)$; in fact, if $\big(P_A(\lambda)f, f\big) = 0$, then $\big((\mathcal{C}^A - \lambda)\mathbf{x}, \mathbf{x}\big) = -\lambda^{n-1}\big(P_A(\lambda)f, f\big) = 0$ for $\mathbf{x} := (f, \lambda f, \dots, \lambda^{n-1} f)^{\mathrm{t}}$.

The next theorem shows that $W(P_A)$ is even contained in the block numerical range of its companion operator $\mathcal{C}^A$:

**Theorem 1.12.1** $W(P_A) \subset W^n\big(\mathcal{C}^A\big)$.

***Proof.*** Let $\lambda_0 \in W(P_A)$. Then there exists an $x \in \mathcal{H}_0$, $\|x\| = 1$, such that $\lambda_0$ is a zero of the scalar polynomial

$$\big(P_A(\lambda)x, x\big) = \lambda^n + \lambda^{n-1}(A_{n-1}x, x) + \dots + \lambda(A_1 x, x) + (A_0 x, x) = 0.$$

The companion operator of the scalar polynomial $\big(P_A(\lambda)x, x\big)$ is the $n \times n$ matrix $\mathcal{C}^A_{(x,\dots,x)}$. Since the zeroes of $\big(P_A(\lambda)x, x\big)$ coincide with the eigenvalues of $\mathcal{C}^A_{(x,\dots,x)}$, it follows that $\lambda_0 \in \sigma_{\mathrm{p}}\big(\mathcal{C}^A_{(x,\dots,x)}\big) \subset W^n\big(\mathcal{C}^A\big)$. □

**Example 1.12.2** To illustrate Theorem 1.12.1, we reconsider the matrix $\mathcal{A}_3$ in Example 1.3.3. It is the companion operator of the quadratic matrix polynomial

$$P_3(\lambda) := \lambda^2 I_2 + \lambda \begin{pmatrix} -\mathrm{i} & -5\mathrm{i} \\ 5\mathrm{i} & -\mathrm{i} \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

Figure 1.12 shows the numerical range of $P_3$ on the left. It is contained in the quadratic numerical range of $\mathcal{A}_3$ with respect to the decomposition $\mathbb{C}^2 \oplus \mathbb{C}^2$ on the right.

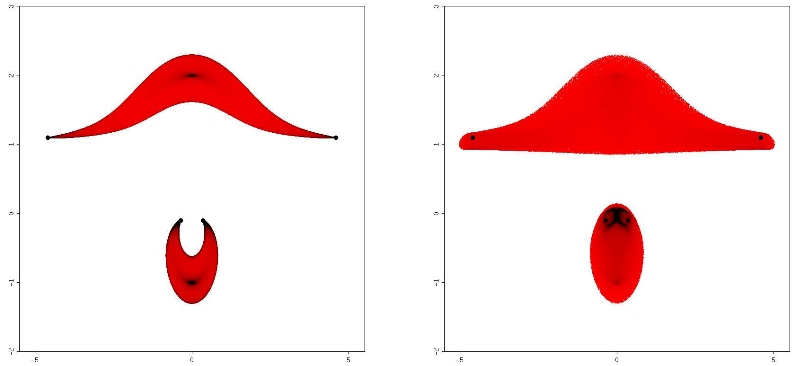


Figure 1.12 $W(P_3)$ and $W_{\mathbb{C}^2 \oplus \mathbb{C}^2}(\mathcal{A}_3)$.

Next we show that if $\mathcal{H}_0$ is finite-dimensional, $\mathcal{H}_0 = \mathbb{C}^k$, then, up to the point 0, the numerical range of $P_A$ coincides with a higher degree block numerical range of its companion operator. To this end, we consider $\mathcal{C}^A$ with respect to a refined decomposition of $\mathcal{H} = \mathcal{H}_0^n = \mathbb{C}^{nk}$.

**Theorem 1.12.3** *If we consider the companion operator $\mathcal{C}^A$ with respect to the decomposition*

$$\mathbb{C}^{nk} = \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{(n-1)k} \oplus \mathbb{C}^k, \tag{1.12.1}$$

*then*

$$W_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}\oplus\mathbb{C}^k}\big(\mathcal{C}^A\big) = W^{(n-1)k+1}\big(\mathcal{C}^A\big) = \begin{cases} W(P_A), & n=1, \\ W(P_A)\cup\{0\}, & n>1. \end{cases}$$

***Proof.*** For $n=1$ the assertion is immediate. For $n>1$, with respect to the decomposition (1.12.1), $\mathcal{C}^A$ has the block operator representation

$$\begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1,k} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & & 0 & \cdots & 0 & 0_{1,k} \\
 & \vdots & & & \vdots & & & \vdots & & \ddots & & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 0 & 0_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 1 & 0_{1,k} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & e_1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & e_k \\
-A_0^{(1)} & \cdots & -A_0^{(k)} & -A_1^{(1)} & \cdots & -A_1^{(k)} & -A_2^{(1)} & \cdots & -A_2^{(k)} & \cdots & -A_{n-2}^{(1)} & \cdots & -A_{n-2}^{(k)} & -A_{n-1}
\end{pmatrix}.$$

Here $0_{1,k} = (0\cdots0) \in L(\mathbb{C}^k,\mathbb{C})$ is the zero vector, $e_j = (0\cdots1\cdots0) \in L(\mathbb{C}^k,\mathbb{C})$ is the $j$-th row unit vector, $j=1,\dots,k$, and

$$A_i^{(j)} = \begin{pmatrix} a_{1j}^{(i)} \\ \vdots \\ a_{kj}^{(i)} \end{pmatrix} \in L(\mathbb{C},\mathbb{C}^k)$$

is the $j$-th column of $A_i = \big(a_{st}^{(i)}\big)_{s,t=1}^k$, $i=0,\dots,n-1$, $j=1,\dots,k$. Now let

$$x := \Big(x_0^{(1)} \dots\, x_0^{(k)} \dots\, x_{n-2}^{(1)} \dots\, x_{n-2}^{(k)}\ (\xi_1 \dots \xi_k)\Big)^{\mathrm{t}} \in \mathbb{C}\oplus\cdots\oplus\mathbb{C}\oplus\mathbb{C}^k$$

with $\big|x_0^{(1)}\big| = \cdots = \big|x_{n-2}^{(k)}\big| = \|\xi\| = 1$ where $\xi := (\xi_1 \dots \xi_k)^{\mathrm{t}}$. By similar manipulations of determinants as in the proof of the previous theorem, one can show that, for $\lambda \neq 0$,

$$\det\big(\mathcal{C}_x^A - \lambda\big) = (-1)^n \lambda^{(n-1)(k-1)}\big(P_A(\lambda)\xi,\xi\big).$$

This implies $W^{(n-1)k+1}(\mathcal{C}^A) \setminus \{0\} = W(P_A) \setminus \{0\}$. Finally, it is not difficult to see that $0 \in W^{(n-1)k+1}(\mathcal{C}^A)$ for $n > 1$. $\square$

**Example 1.12.4** As an example for Theorem 1.12.3, we consider the quadratic matrix polynomial (compare [LMZ98])

$$P_{11}(\lambda) := \lambda^2 I_2 + \lambda \begin{pmatrix} 0 & 2.8\mathrm{i} \\ -2.8\mathrm{i} & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \lambda \in \mathbb{C},$$

with its companion operator $\mathcal{A}_{11}$ decomposed as

$$\mathcal{A}_{11} = \left( \begin{array}{c|c|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -2 & -1 & 0 & -2.8\mathrm{i} \\ -1 & -2 & 2.8\mathrm{i} & 0 \end{array} \right).$$

In Fig. 1.13 the cubic numerical range of $\mathcal{A}_{11}$ with respect to this decomposition and the numerical range of the operator polynomial $P_{11}$ are displayed. A closer look shows that the numerical range of $P_{11}$ on the right does not contain the point 0, whereas the cubic numerical range on the left does.

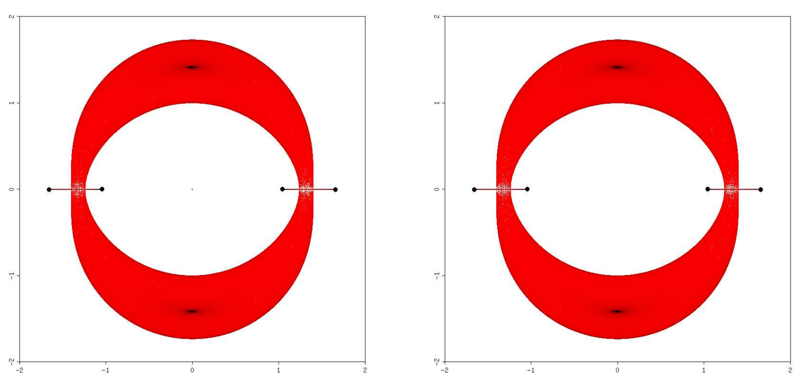

Figure 1.13 $W_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^2}(\mathcal{A}_{11})$ and $W(P_{11})$.

**Remark 1.12.5** In [Lin03] H. Linden applied the quadratic numerical range to derive enclosures for the zeroes of monic polynomials in $\mathbb{C}$ of degree $n \geq 3$. He enclosed the quadratic numerical range of the companion matrix $\mathcal{C}_A$ with respect to the decompositions $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$ and $\mathbb{C}^n = \mathbb{C}^{n-2} \oplus \mathbb{C}^2$ (and thus the zeroes) in two circles of equal radius. Examples show that the enclosures are tighter than those obtained from the numerical range of $\mathcal{C}_A$.

## 1.13 Gershgorin's theorem for block operator matrices

Gershgorin's circle theorem is a valuable tool to enclose the spectrum of matrices (see [Ger31], [Bra58], [Var04]). Its generalization to partitioned matrices and bounded block operator matrices (see [FV62], [Sal99]) is straightforward. In general, there is no inclusion between the quadratic or block numerical range and the Gershgorin sets; depending on the particular situation, one or the other may give a better spectral enclosure. However, the quadratic or, more generally, block numerical range has the advantage of not using norms of inverses.

The following Gershgorin theorem for bounded block operator matrices is due to H. Salas (see [Sal99]). Its proof generalizes Householder's proof of Gershgorin's theorem in the matrix case; it may even be generalized to unbounded diagonally dominant block operator matrices.

**Theorem 1.13.1** *Let $n \in \mathbb{N}$, let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be complex Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, and let $\mathcal{A} \in L(\mathcal{H})$,*

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

*with $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$, $i, j = 1, \dots, n$. If we define*

$$\mathcal{G}_i := \sigma(A_{ii}) \cup \Big\{ \lambda \in \rho(A_{ii}) : \big\| (A_{ii} - \lambda)^{-1} \big\|^{-1} \le \sum_{\substack{j=1 \\ j \ne i}}^{n} \big\| A_{ij} \big\| \Big\} \qquad (1.13.1)$$

*for $i = 1, \dots, n$, then*

$$\sigma(\mathcal{A}) \subset \bigcup_{i=1}^{n} \mathcal{G}_i .$$

***Proof.*** Suppose that $\lambda \notin \bigcup_{i=1}^n \sigma(A_{ii})$. Then we can write

$$\mathcal{A} - \lambda = \begin{pmatrix} A_{11} - \lambda & & 0 \\ & \ddots & \\ 0 & & A_{nn} - \lambda \end{pmatrix} \big( I + M(\lambda) \big) \qquad (1.13.2)$$

where

$$M(\lambda) := \begin{pmatrix} 0 & (A_{11} - \lambda)^{-1} A_{12} & \cdots & (A_{11} - \lambda)^{-1} A_{1n} \\ (A_{22} - \lambda)^{-1} A_{21} & 0 & & (A_{22} - \lambda)^{-1} A_{2n} \\ \vdots & & \ddots & \vdots \\ (A_{nn} - \lambda)^{-1} A_{n1} & \cdots & \cdots & 0 \end{pmatrix}.$$

If $\|(A_{ii}-\lambda)^{-1}\|^{-1} > \sum_{\substack{j=1\\j\neq i}}^{n} \|A_{ij}\|$ for $i=1,\dots,n$, then $\|M(\lambda)\|<1$. Hence both factors in (1.13.2) are boundedly invertible and so $\lambda\in\rho(\mathcal{A})$. □

**Remark 1.13.2** For a self-adjoint diagonal entry $A_{ii}$, the norms of the inverses in (1.13.1) are known explicitly and we have

$$\mathcal{G}_i = \sigma(A_{ii}) \cup \Big\{\lambda\in\rho(A_{ii}) : \operatorname{dist}\big(\lambda,\sigma(A_{ii})\big) \le \sum_{\substack{j=1\\j\neq i}}^{n}\|A_{ij}\|\Big\};$$

for non-self-adjoint $A_{ii}$, only the estimate in terms of the numerical range is available and thus, in general, we only have the inclusion

$$\mathcal{G}_i \subset \sigma(A_{ii}) \cup \Big\{\lambda\in\rho(A_{ii}) : \operatorname{dist}\big(\lambda,W(A_{ii})\big) \le \sum_{\substack{j=1\\j\neq i}}^{n}\|A_{ij}\|\Big\}.$$

In the remaining part of this section we consider the case $n=2$. For a $2\times 2$ block operator matrix $\mathcal{A}$, the spectrum of $\mathcal{A}$ can be described in terms of the spectra of the Schur complements as (see Proposition 1.6.2)

$$\sigma(\mathcal{A})\setminus\sigma(A_{ii}) = \sigma(S_i), \quad i=1,2. \tag{1.13.3}$$

This description and a spectral enclosure for the Schur complements allows us to tighten the spectral enclosure (1.13.1) by the Gershgorin sets:

**Proposition 1.13.3** *Let $n=2$ and define*

$$\mathcal{N}_1 := \sigma(A_{11})\cup\Big\{\lambda\in\rho(A_{11})\cap\rho(A_{22}) : \big\|(A_{11}-\lambda)^{-1}A_{12}(A_{22}-\lambda)^{-1}A_{21}\big\|\ge 1\Big\},$$
$$\mathcal{N}_2 := \sigma(A_{22})\cup\Big\{\lambda\in\rho(A_{11})\cap\rho(A_{22}) : \big\|(A_{22}-\lambda)^{-1}A_{21}(A_{11}-\lambda)^{-1}A_{12}\big\|\ge 1\Big\}.$$

*Then*

$$\sigma(\mathcal{A}) \subset \big(\sigma(A_{11})\cup\sigma(S_1)\big)\cup\big(\sigma(A_{22})\cup\sigma(S_2)\big) \subset \mathcal{N}_1\cup\mathcal{N}_2 \subset \mathcal{G}_1\cup\mathcal{G}_2. \tag{1.13.4}$$

***Proof.*** The first inclusion in (1.13.4) is obvious from (1.13.3). For the second inclusion, we observe that we have $\lambda\notin\mathcal{N}_1\cup\mathcal{N}_2$ if and only if $\lambda\in\rho(A_{11})\cap\rho(A_{22})$ and the two inequalities

$$\big\|(A_{11}-\lambda)^{-1}A_{12}(A_{22}-\lambda)^{-1}A_{21}\big\| < 1, \tag{1.13.5}$$
$$\big\|(A_{22}-\lambda)^{-1}A_{21}(A_{11}-\lambda)^{-1}A_{12}\big\| < 1 \tag{1.13.6}$$

hold. Since, for $\lambda\in\rho(A_{11})\cap\rho(A_{22})$, we can write

$$S_1(\lambda) = (A_{11}-\lambda)\big(I-(A_{11}-\lambda)^{-1}A_{12}(A_{22}-\lambda)^{-1}A_{21}\big),$$
$$S_2(\lambda) = (A_{22}-\lambda)\big(I-(A_{22}-\lambda)^{-1}A_{21}(A_{11}-\lambda)^{-1}A_{12}\big),$$

we conclude that $\lambda \notin \mathcal{N}_1 \cup \mathcal{N}_2$ implies $\lambda \in \rho(S_1) \cap \rho(S_2)$. Because $S_i$ is defined on $\mathbb{C} \setminus \sigma(A_{ii})$, we have $\rho(S_i) \,\dot\cup\, \sigma(S_i) = \mathbb{C} \setminus \sigma(A_{ii})$, $i = 1, 2$. Hence $\lambda \in \rho(S_1) \cap \rho(S_2)$ is equivalent to $\lambda \notin \big(\sigma(A_{11}) \cup \sigma(S_1)\big) \cup \big(\sigma(A_{22}) \cup \sigma(S_2)\big)$.

For the third inclusion in (1.13.4), we note that $\lambda \notin \mathcal{G}_1 \cup \mathcal{G}_2$ if and only if $\lambda \notin \sigma(A_{11}) \cup \sigma(A_{22})$ and the two inequalities

$$\big\|(A_{11} - \lambda)^{-1}\big\| \, \|A_{12}\| < 1, \quad \big\|(A_{22} - \lambda)^{-1}\big\| \, \|A_{21}\| < 1$$

hold; the latter imply the two inequalities (1.13.5), (1.13.6). Therefore $\lambda \notin \mathcal{G}_1 \cup \mathcal{G}_2$ implies $\lambda \notin \mathcal{N}_1 \cup \mathcal{N}_2$. □

To conclude this section, we compare the spectral enclosures by the Gershgorin sets to those by the quadratic numerical range for self-adjoint and $\mathcal{J}$-self-adjoint $2 \times 2$ block operator matrices. In particular, we consider situations where the quadratic numerical range yields estimates of the spectrum that are independent of the size of the off-diagonal entries.

**Remark 1.13.4** Let $n = 2$, $A_{11} = A_{11}^*$, $A_{22} = A_{22}^*$, and suppose that either $A_{21} = A_{12}^*$ or $A_{21} = -A_{12}^*$.

i) The Gershgorin type Theorem 1.13.1 yields the inclusion

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A_{11}) \cup \sigma(A_{22})\big) \le \|A_{12}\|\big\}.$$

ii) If $\operatorname{dist}\big(\sigma(A_{11}), \sigma(A_{22})\big) > 0$ and $A_{21} = A_{12}^*$, then Theorem 1.3.7 i), which uses the quadratic numerical range, gives the tighter inclusion

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A_{11}) \cup \sigma(A_{22})\big) \le \|\delta_{A_{12}}\|\big\},$$

where

$$\delta_{A_{12}} = \|A_{12}\| \tan\left(\frac{1}{2}\arctan\left(\frac{2\|A_{12}\|}{\operatorname{dist}\big(\sigma(A_{11}), \sigma(A_{22})\big)}\right)\right) < \|A_{12}\|;$$

if, in addition, $\max \sigma(A_{22}) < \min \sigma(A_{11})$, then, by Theorem 1.3.6 ii),

$$\sigma(\mathcal{A}) \cap \big(\max \sigma(A_{22}), \min \sigma(A_{11})\big) = \emptyset$$

independently of the size of $\|\mathcal{A}_{12}\|$.

iii) If $A_{21} = -A_{12}^*$, then Proposition 1.3.9 i) shows that

$$\operatorname{Re} \sigma(\mathcal{A}) \subset \big[\min\big\{\min \sigma(A_{11}), \min \sigma(A_{22})\big\}, \max\big\{\max \sigma(A_{11}), \max \sigma(A_{22})\big\}\big]$$

independently of the size of $\|\mathcal{A}_{12}\|$.

If the spectra of the diagonal elements are not disjoint, then the Gershgorin enclosure in Remark 1.13.4 i) still applies, but the estimates in Remark 1.13.4 ii) do not. The following example shows that, in some cases, a change of the decomposition of the space leads to a block operator matrix with diagonal elements having disjoint spectra.

**Example 1.13.5** Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$, and let $A_{11} = A_{11}^*$, $A_{22} = A_{11}$, $A_{21} = A_{12}^*$ be such that $\sigma(A_{11}) = \sigma_1 \dot\cup \sigma_2$ with $\sigma_i \neq \emptyset$, $i = 1, 2$, and $\operatorname{dist}(\sigma_1, \sigma_2) > 2\,\|A_{12}\|$. Then the two Gershgorin sets $\mathcal{G}_1, \mathcal{G}_2$ coincide and consist of two components,

$$\mathcal{G}_1 = \mathcal{G}_2 = \Sigma_1 \dot\cup \Sigma_2, \quad \Sigma_i := \big\{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma_i) \leq \|A_{12}\|\big\}, \quad i = 1, 2,$$

whereas the quadratic numerical range with respect to the given decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ consists of a single component by Corollary 1.1.10 ii).

The inclusion by the quadratic numerical range may be improved by using another decomposition of $\mathcal{H}$. Since $\sigma(A_{11}) = \sigma_1 \dot\cup \sigma_2$, there exist invariant subspaces $\mathcal{H}_1^1$, $\mathcal{H}_1^2$ of $A_{11}$ such that

$$\sigma\big(A_{11}\big|_{\mathcal{H}_1^1}\big) = \sigma_1, \quad \sigma\big(A_{11}\big|_{\mathcal{H}_1^2}\big) = \sigma_2.$$

We consider the new decomposition $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$ with $\widetilde{\mathcal{H}}_i = \mathcal{H}_1^i \oplus \mathcal{H}_1^i$ for $i = 1, 2$. By a standard perturbation argument for self-adjoint operators (see [Kat95, Theorem V.4.10]), the assumption $\operatorname{dist}(\sigma_1, \sigma_2) > 2\,\|A_{12}\|$ implies that the new diagonal elements $\widetilde{A}_{ii}$, $i = 1, 2$, have separated spectra (they are contained *e.g.* in the two disjoint components $\Sigma_i$ of the Gershgorin sets). Hence Remark 1.13.4 ii) applies to the block operator matrix obtained with respect to this new decomposition.

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## Chapter 2

# Unbounded Block Operator Matrices

Unbounded block operator matrices provide an efficient way to describe coupled systems of partial differential equations of mixed order and type. Hence information about the spectral properties of block operator matrices is of major interest. We distinguish three classes of block operator matrices, depending on the position of the dominating operators: diagonally dominant, off-diagonally dominant, and upper dominant. The main results concern the localization and structure of the spectrum, variational principles and estimates for eigenvalues in gaps of the spectrum, criteria for block diagonalizability as well as existence and uniqueness of solutions of Riccati equations.

### 2.1 Relative boundedness and relative compactness

In this chapter the perturbation theory of unbounded linear operators plays a crucial role. Here we provide some general results on relatively bounded and relatively compact perturbations. More specific perturbation results are stated within the sections where they are used.

Let $E, F$ be Banach spaces. A linear operator $T$ from $E$ to $F$ with domain $\mathcal{D}(T)$ is called *closed* if its graph $\mathcal{G}(T) := \big\{(x, Tx) : x \in \mathcal{D}(T)\big\}$ is a closed subspace of $E \times F$ and *closable* if the closure $\overline{\mathcal{G}(T)}$ of its graph is a graph; in this case, the operator $\overline{T}$ with $\overline{\mathcal{G}(T)} = \mathcal{G}\big(\overline{T}\big)$ is called the *closure* of $T$. If $T$ is closed and $S$ is a closable linear operator with domain $\mathcal{D}(S)$ such that $\overline{S} = T$, then $\mathcal{D}(S)$ is called *core* of $T$ (see [Kat95, Section III.5]). Note that a bounded linear operator $T$ is closed if and only if its domain $\mathcal{D}(T)$ is closed; in particular, an everywhere defined bounded linear operator is closed. Conversely, the closed graph theorem says that an everywhere defined closed linear operator is bounded (see [Kat95, Theorem III.5.20]).

**Definition 2.1.1** Let $T$ be a closable linear operator in a Banach space $E$. The *resolvent set* and the *spectrum* of $T$ are defined as

$$\rho(T) := \big\{\lambda \in \mathbb{C} : T-\lambda \text{ is injective}, \, (T-\lambda)^{-1} \in L(E)\big\},$$
$$\sigma(T) := \mathbb{C} \setminus \rho(T),$$

and the *point spectrum*, *continuous spectrum*, and *residual spectrum* as

$$\sigma_{\mathrm{p}}(T) := \big\{\lambda \in \mathbb{C} : T-\lambda \text{ is not injective}\big\},$$
$$\sigma_{\mathrm{c}}(T) := \big\{\lambda \in \mathbb{C} : T-\lambda \text{ is injective}, \, \overline{R(T-\lambda)} = E, \, R(T-\lambda) \neq E\big\},$$
$$\sigma_{\mathrm{r}}(T) := \big\{\lambda \in \mathbb{C} : T-\lambda \text{ is injective}, \, \overline{R(T-\lambda)} \neq E\big\}.$$

Note that $\rho(T) \neq \emptyset$ implies that $T$ is closed; in fact, if $\lambda \in \rho(T)$, then $(T-\lambda)^{-1}$ is closed and hence so is $T-\lambda$. Then, by the closed graph theorem,

$$\rho(T) = \big\{\lambda \in \mathbb{C} : T-\lambda \text{ is bijective}\big\}$$

and hence $\sigma(T) = \sigma_{\mathrm{p}}(T) \,\dot{\cup}\, \sigma_{\mathrm{c}}(T) \,\dot{\cup}\, \sigma_{\mathrm{r}}(T)$ (see [EE87, Section I.1]).

**Definition 2.1.2** Let $E$, $F$, $G$ be Banach spaces and let $T, S$ be linear operators from $E$ to $F$ and from $E$ to $G$, respectively.

i) $S$ is called *relatively bounded* with respect to $T$ (or $T$*-bounded*) if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist constants $a_S$, $b_S \geq 0$ such that
$$\|Sx\| \leq a_S\|x\| + b_S\|Tx\|, \quad x \in \mathcal{D}(T). \tag{2.1.1}$$
The infimum $\delta_S$ of all $b_S$ so that (2.1.1) holds for some $a_S \geq 0$ is called *relative bound* of $S$ with respect to $T$ (or $T$*-bound* of $S$, see [Kat95, Section IV.1.1]).

ii) $S$ is called *relatively compact* with respect to $T$ (or $T$*-compact*) if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and, for every bounded sequence $(x_n)_1^\infty \subset \mathcal{D}(T)$ such that $(Tx_n)_1^\infty \subset F$ is bounded, the sequence $(Sx_n)_1^\infty \subset G$ contains a convergent subsequence (see [Kat95, Section IV.1.3]).

**Remark 2.1.3** The following observations are useful:

i) If $T$ is closed and $S$ is closable, then $\mathcal{D}(T) \subset \mathcal{D}(S)$ already implies that $S$ is $T$-bounded (see [Kat95, Remark IV.1.5]).

ii) The inequality (2.1.1) is equivalent to
$$\|Sx\|^2 \leq a_S'^2\|x\|^2 + b_S'^2\|Tx\|^2, \quad x \in \mathcal{D}(T), \tag{2.1.2}$$
with $a_S', b_S' \geq 0$; moreover, (2.1.1) holds with $b_S < \delta$ for some $\delta > 0$ if and only if (2.1.2) holds with $b_S' < \delta$ (see [Kat95, Section V.4.1, (4.1), (4.2)]). This follows from the simple fact that, for arbitrary $\gamma > 0$,
$$\big(\xi_1 + \xi_2\big)^2 \leq \big(1+\gamma^{-1}\big)\,\xi_1^2 + (1+\gamma)\,\xi_2^2, \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{2.1.3}$$

In general, the sum of closable or closed operators is not closable or closed, respectively. However, closability and closedness are stable under relatively bounded perturbations with relative bound $< 1$. For the stability of bounded invertibility, an additional condition is required.

**Theorem 2.1.4** *Let $E$, $F$ be Banach spaces, let $T, S$ be linear operators from $E$ to $F$, and let $S$ be $T$-bounded with $T$-bound $< 1$.*

i) *$T{+}S$ is closable if and only if so is $T$; then $\mathcal{D}\big(\overline{T{+}S}\big){=}\mathcal{D}\big(\overline{T}\big)$. In particular, $T{+}S$ is closed if and only if so is $T$ (see* [Kat95, Theorem IV.1.1]*).*
ii) *$T + S$ is boundedly invertible if so is $T$ and the constants $a_S$, $b_S$ in inequality* (2.1.1) *satisfy*
$$a_S\|T^{-1}\| + b_S < 1$$
(*see* [Kat95, Theorem IV.1.16]).

**Corollary 2.1.5** *Let $E$, $F$ be Banach spaces and let $T, S$ be linear operators from $E$ to $F$. Suppose there exists a ray $\Theta_{\rho,\varphi} := \{r\mathrm{e}^{\mathrm{i}\varphi} : r \geq \rho\}$ with $\rho \geq 0$, $\varphi \in (-\pi, \pi]$ and a constant $M \geq 0$ such that $\Theta_{\rho,\varphi} \subset \rho(T)$ and*
$$\big\|(T-\lambda)^{-1}\big\| \leq \frac{M}{|\lambda|}, \quad \lambda \in \Theta_{\rho,\varphi}. \tag{2.1.4}$$
*If $S$ is $T$-bounded with $T$-bound $< 1/(M+1)$, then there exists an $R \geq \rho$ such that $\Theta_{R,\varphi} \subset \rho(T+S)$.*

***Proof.*** Let $\lambda \in \Theta_{\rho,\varphi}$. There exist $a_S, b_S \geq 0$, $b_S < 1/(M+1)$, such that
$$\|Sx\| \leq a_S\|x\| + b_S\|Tx\| \leq \big(a_S + b_S|\lambda|\big)\|x\| + b_S\|(T-\lambda)x\|, \quad x \in \mathcal{D}(T).$$
Theorem 2.1.4 ii), applied to $T - \lambda$ and $S$, shows that $\lambda \in \rho(T+S)$ if
$$\big(a_S + b_S|\lambda|\big)\,\big\|(T-\lambda)^{-1}\big\| + b_S < 1.$$
Due to assumption (2.1.4), this inequality is satisfied if
$$a_S\frac{M}{|\lambda|} + (1+M)\,b_S < 1.$$
The latter holds if $|\lambda| \geq R$ with $R \geq \rho$ such that $R > \dfrac{a_S M}{1-(1+M)b_S}$. □

**Lemma 2.1.6** *If $S$ is $T$-bounded with $T$-bound $\delta < 1$, then $S$ is $(T+S)$-bounded with $(T+S)$-bound $\leq \delta/(1-\delta)$.*

***Proof.*** By assumption, there exist $a_S$, $b_S \geq 0$, $\delta \leq b_S < 1$, such that
$$\|Sx\| \leq a_S\|x\| + b_S\|Tx\| \leq a_S\|x\| + b_S\|(T+S)x\| + b_S\|Sx\|, \quad x \in \mathcal{D}(T).$$
Since $b_S < 1$, it follows that
$$\|Sx\| \leq \frac{a_S}{1-b_S}\|x\| + \frac{b_S}{1-b_S}\|(T+S)x\|, \quad x \in \mathcal{D}(T).$$
□

If $T$ is closed, then $D_T = \big(\mathcal{D}(T), \|\cdot\|_T\big)$ with the graph norm $\|x\|_T := \|x\| + \|Tx\|$, $x \in \mathcal{D}(T)$, is a Banach space. Obviously, $S$ is $T$-bounded if and only if $S$ is a bounded operator from $D_T$ to $G$, and $S$ is $T$-compact if and only if $S$ is compact from $D_T$ to $G$. Hence every relatively compact operator is relatively bounded; in fact, much more can be said.

**Proposition 2.1.7** *Let $E$, $F$ be reflexive Banach spaces, let $T, S$ be linear operators from $E$ to $F$ such that $T$ or $S$ is closable. If $S$ is $T$-compact, then $S$ is $T$-bounded with $T$-bound* 0 (*see* [EE87, Corollary III.7.7]).

**Lemma 2.1.8** *Let $E$, $F$, $G$ be Banach spaces and let $T, S$ be linear operators from $E$ to $F$ and from $E$ to $G$, respectively.*

i) *If $T = T_0 + T_1$, $T_1$ is $T_0$-bounded with $T_0$-bound $< 1$, and $S$ is $T_0$-compact, then $S$ is $T$-compact.*

ii) *Let $S = S_0 + S_1$ be $T$-bounded. If $S_1$ is $S_0$-bounded with $S_0$-bound $< 1$, then $S_0$ is $T$-bounded; if $S_1$ is $S_0$-compact, $E$, $G$ are reflexive, and $S_0$ or $S_1$ is closable, then $S_0$ is $T$-bounded and $S_1$ is $T$-compact.*

***Proof.*** i) Let $(x_n)_1^\infty \subset \mathcal{D}(T)$ be a sequence such that $(x_n)_1^\infty$ and $(Tx_n)_1^\infty$ are bounded. By Lemma 2.1.6, $T_1$ is $T$-bounded; hence $(T_1 x_n)_1^\infty$ is bounded and so is $(T_0 x_n)_1^\infty = (Tx_n - T_1 x_n)_1^\infty$. Since $S$ is $T_0$-compact, it follows that there exists a subsequence $(x_{n_k})_1^\infty \subset (x_n)_1^\infty$ such that $(Sx_{n_k})_1^\infty$ converges.

ii) By Lemma 2.1.6, the first assumption implies that $S_1$ is $S$-bounded; since $S$ is, in turn, $T$-bounded, it follows that $S_1$ is $T$-bounded and hence so is $S_0 = S - S_1$. Now let $S_1$ be $S_0$-compact and let $(x_n)_1^\infty \subset \mathcal{D}(T)$ be a sequence such that $(x_n)_1^\infty$ and $(Tx_n)_1^\infty$ are bounded. By Proposition 2.1.7, $S_1$ is $S_0$-bounded with $S_0$-bound 0. Due to the first claim in ii), $S_0$ is $T$-bounded and so $(S_0 x_n)_1^\infty$ is bounded. Because $S_1$ is $S_0$-compact, there exists a subsequence $(x_{n_k})_1^\infty \subset (x_n)_1^\infty$ so that $(S_1 x_{n_k})_1^\infty$ converges. □

Relatively compact perturbations have some useful properties, *e.g.* in view of adjoints $(T+S)^*$ of densely defined sums of unbounded operators or in view of the essential spectrum. First we need the notion of Fredholm operators (see [GGK90, Chapter XVII.2] and [EE87, Chapter IX.1, $k = 3$]).

**Definition 2.1.9** A closed linear operator $T$ in a Banach space $E$ is called *Fredholm* if for its kernel $\ker T$ and its range $R(T)$

$$n(T) := \dim \ker T < \infty, \quad d(T) := \dim\big(E/R(T)\big) < \infty;$$

in this case $\operatorname{ind}(T) := n(T) - d(T)$ is called the *index* of $T$. The *essential spectrum* of $T$ is defined as

$$\sigma_{\mathrm{ess}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

Note that the condition $d(T) < \infty$ automatically implies that the range of a Fredholm operator is closed (see [GGK90, Corollary XI.2.3]).

For non-self-adjoint linear operators, there are several other definitions for the essential spectrum (see *e.g.* [EE87, Section IX.1, $k = 1, 2, 4, 5$]). The definition used here yields the following information about the complement $\sigma(T) \setminus \sigma_{\text{ess}}(T)$.

**Theorem 2.1.10** *Let $T$ be a linear operator, $\rho(T) \neq \emptyset$, and $\Omega$ an open connected subset of $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$. If $\Omega \cap \rho(T) \neq \emptyset$, then $\sigma(T) \cap \Omega$ is discrete, i.e. consists of countably many isolated eigenvalues of finite algebraic multiplicity with no accumulation point in $\Omega$ (see* [GGK90, Theorem XVII.2.1]*).*

For the adjoint of a sum of unbounded operators, in general, only the inclusion $(T+S)^* \supset T^* + S^*$ holds. If $S$ is bounded, equality prevails. A more general result is due to R.W. Beals (see also [Gol66, Corollary V.3.9]):

**Proposition 2.1.11** *Let $E$, $F$ be Banach or Hilbert spaces and let $T, S$ be densely defined linear operators from $E$ to $F$. If $T$ is Fredholm, $S$ is $T$-compact, and $S^*$ is $T^*$-compact, then $(T + S)^* = T^* + S^*$ (see* [Bea64]*).*

**Remark 2.1.12** If $S$ is $T$-compact, then $S^*$ need not be $T^*$-compact. In fact, it may happen that $\mathcal{D}(T^*) \cap \mathcal{D}(S^*) = \{0\}$ even if $T$ is the inverse of a positive definite compact operator in a Hilbert space (see [Bea64]).

It is a classical result of Weyl that relatively compact perturbations of a self-adjoint operator do not alter the essential spectrum (see [Wey09]). Here we use the following two generalizations for closed linear operators (see [EE87, Theorem IX.2.1, $k = 3$] and [Kat95, Theorem IV.5.35, footnote 1]).

**Theorem 2.1.13** *Let $E$ be a Banach space. If $T$ is a closed linear operator in $E$ and $S$ is a $T$-compact operator in $E$, then*

$$\sigma_{\text{ess}}(T + S) = \sigma_{\text{ess}}(T),$$

*and* $\operatorname{ind}(T - \lambda) = \operatorname{ind}(T + S - \lambda)$ *for $\lambda \notin \sigma_{\text{ess}}(T)$.*

**Remark 2.1.14** Theorem 2.1.13 also holds if one uses the definitions of the essential spectrum from [EE87, Section IX.1] for $k = 1, 2, 4$, but not for $k = 5$; in this case, additional hypotheses are required.

A weaker criterion for the stability of the essential spectrum is the compactness of the difference of the resolvents.

**Theorem 2.1.15** *Let $E$ be a Banach space and let $T$, $T_0$ be closed linear operators in $E$ with $\rho(T) \cap \rho(T_0) \neq \emptyset$. If, for some $\lambda_0 \in \rho(T) \cap \rho(T_0)$, the difference $(T - \lambda_0)^{-1} - (T_0 - \lambda_0)^{-1}$ is compact, then*

$$\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T_0),$$

*and* $\text{ind}(T-\lambda) = \text{ind}(T_0-\lambda)$ *for* $\lambda \notin \sigma_{\text{ess}}(T)$.

***Proof.*** Let $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$. The proof of this theorem is based on the relation

$$T-\lambda = (\lambda-\lambda_0)(T-\lambda_0)\left((\lambda-\lambda_0)^{-1}-(T-\lambda_0)^{-1}\right),$$

and an analogous formula for $T_0$. By the spectral mapping theorem for the essential spectrum (see [EE87, Theorem IX.2.3, $k=3$]) and the compactness assumption, we have

$$\begin{aligned} \lambda \in \sigma_{\text{ess}}(T) &\iff (\lambda-\lambda_0)^{-1} \in \sigma_{\text{ess}}\big((T-\lambda_0)^{-1}\big) = \sigma_{\text{ess}}\big((T_0-\lambda_0)^{-1}\big) \\ &\iff \lambda \in \sigma_{\text{ess}}(T_0). \end{aligned}$$

Since $\lambda_0 \in \rho(T)\cap\rho(T_0)$, the operators $T-\lambda_0$ and $T_0-\lambda_0$ are Fredholm with index 0. Then, for $\lambda \notin \sigma_{\text{ess}}(T)$, the index stability theorem (see [GGK90, Theorem XVII.3.1]) yields that

$$\begin{aligned} \text{ind}(T-\lambda) &= \text{ind}\left((T-\lambda_0)^{-1}-(\lambda-\lambda_0)^{-1}\right) \\ &= \text{ind}\left((T_0-\lambda_0)^{-1}-(\lambda-\lambda_0)^{-1}\right) = \text{ind}(T_0-\lambda). \qquad \square \end{aligned}$$

**Remark 2.1.16** The assumption of Theorem 2.1.13 implies the one in Theorem 2.1.15. In fact, if $S$ is $T$-compact and $\lambda_0 \in \rho(T)\cap\rho(T+S)$, then $S(T-\lambda_0)^{-1}$ is compact (see [Wei00, Satz 9.12]) and hence, by the second resolvent identity, so is

$$(T+S-\lambda_0)^{-1}-(T-\lambda_0)^{-1} = -(T+S-\lambda_0)^{-1}S\,(T-\lambda_0)^{-1}.$$

Domain inclusions, in general, only imply relative boundedness (see Remark 2.1.3 i)). If the unperturbed operator has some additional properties which allow to define fractional powers, much more can be said.

**Definition 2.1.17** For $\omega \in [0,\pi)$ we define the sector

$$\Sigma_\omega := \left\{ r\mathrm{e}^{\mathrm{i}\phi} : r \ge 0,\ |\phi| \le \omega \right\} \subset \mathbb{C}. \tag{2.1.5}$$

A densely defined linear operator $T$ in a Banach space $E$ is called *sectorial* if there exists an $\omega \in [0,\pi)$ such that

(i) $\mathbb{C}\setminus\Sigma_\omega \subset \rho(T)$,
(ii) $\sup_{\lambda\in\mathbb{C}\setminus\Sigma_\omega} \left\|(T-\lambda)^{-1}\right\| < \infty$,

and $T$ is called *m-accretive* if $\omega \le \pi/2$. The infimum of all such $\omega$ is called (*sectoriality*) *angle* of $T$ (see *e.g.* [KW04, Section 9], [Kre71, Chapter I.5.8]).

A sectorial operator $T$ is always closed as $\rho(T) \neq \emptyset$. A Neumann series argument yields the following equivalent characterization of sectoriality.

**Remark 2.1.18** The operator $T$ is sectorial if and only if

(i′) $(-\infty, 0) \subset \rho(T)$,
(ii′) there exists an $M \geq 0$ such that $\big\|(T-\lambda)^{-1}\big\| < \dfrac{M}{|\lambda|}$, $\lambda \in (-\infty, 0)$,

and $T$ is m-accretive if and only if (i′) and (ii′) with $M=1$ hold; in this case,

$$\big\|(T-z)^{-1}\big\| < \frac{1}{|\operatorname{Re} z|}, \quad \operatorname{Re} z < 0. \tag{2.1.6}$$

For a sectorial operator $T$, the fractional powers $T^\gamma$, $0 \leq \gamma \leq 1$, are defined (see *e.g.* [KW04, Section 15], [Kre71, Chapter I.5.8], [Paz83, Section 2.6]); like $T$ itself, they are all densely defined closed linear operators.

**Proposition 2.1.19** *Let $E$, $F$ be Banach spaces, $S$ a closable linear operator from $E$ to $F$, and $T$ a sectorial operator in $E$. If there is a $\gamma \in (0,1)$ with $\mathcal{D}(T^\gamma) \subset \mathcal{D}(S)$, then $S$ is $T$-bounded with $T$-bound 0.*

***Proof.*** Since $T^\gamma$ is closed and $S$ is closable, the inclusion $\mathcal{D}(T^\gamma) \subset \mathcal{D}(S)$ implies that $S$ is $T^\gamma$-bounded. Due to the log-convexity property of the mapping $\alpha \mapsto \|T^\alpha x\|$, there is a $C > 0$ such that, for every $\varepsilon > 0$,

$$\|T^\gamma x\| \leq C\big(\varepsilon^{-\gamma}\|x\| + \varepsilon^{1-\gamma}\|Tx\|\big), \quad x \in \mathcal{D}(T) \subset \mathcal{D}(T^\gamma) \tag{2.1.7}$$

(see [KW04, Theorem 15.14 a)]). This and $\gamma < 1$ imply that $T^\gamma$ is $T$-bounded with $T$-bound 0. Together with the first statement, the claim follows. □

For a non-sectorial operator $T$ in a Hilbert space, Proposition 2.1.19 applies to the sectorial operator $|T|$ and yields:

**Corollary 2.1.20** *Let $E$, $F$ be Hilbert spaces, $S$ a closable linear operator from $E$ to $F$, and $T$ a densely defined closed linear operator in $E$. If there is a $\gamma \in (0,1)$ with $\mathcal{D}(|T|^\gamma) \subset \mathcal{D}(S)$, then $S$ is $T$-bounded with $T$-bound 0.*

***Proof.*** If $(E_\lambda)_{\lambda\in\mathbb{R}}$ is the spectral function of $T$, then the spectral theorem and Hölder's inequality show that, for $x \in \mathcal{D}(T)$,

$$\begin{aligned}\|T^\gamma x\|^2 &= \int_0^\infty \lambda^{2\gamma}\,\mathrm{d}(E_\lambda x, x) \leq \left(\int_0^\infty \mathrm{d}(E_\lambda x, x)\right)^{1-\gamma}\left(\int_0^\infty \lambda^2 \mathrm{d}(E_\lambda x, x)\right)^\gamma \\ &= \|x\|^{2(1-\gamma)}\,\|Tx\|^{2\gamma}.\end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary. Applying Young's inequality, we arrive at

$$\begin{aligned}\|x\|^{1-\gamma}\|Tx\|^\gamma &= \left(\frac{\gamma^{\frac{\gamma}{1-\gamma}}}{\varepsilon^\gamma}\|x\|\right)^{1-\gamma}\left(\frac{\varepsilon^{1-\gamma}}{\gamma}\|Tx\|\right)^\gamma \leq (1-\gamma)\frac{\gamma^{\frac{\gamma}{1-\gamma}}}{\varepsilon^\gamma}\|x\| + \varepsilon^{1-\gamma}\|Tx\| \\ &\leq \varepsilon^{-\gamma}\|x\| + \varepsilon^{1-\gamma}\|Tx\|.\end{aligned}$$

□

**Remark 2.1.21** If $T$ is a nonnegative operator in a Hilbert space $E$, then the inequality (2.1.7) holds with $C = 1$.

For the study of exponentially dichotomous block operator matrices (see Section 2.7), we need a number of variations of the notion of accretivity; here we may specialize to linear operators in Hilbert spaces.

**Definition 2.1.22** Let $E$ be a Hilbert space with scalar product $(\cdot,\cdot)$ and let $T$ be a linear operator in $E$. Denote the *numerical range* of $T$ by

$$W(T) := \big\{(Tx, x) : x \in \mathcal{D}(T),\ \|x\| = 1\big\}. \tag{2.1.8}$$

Then the operator $T$ is called *accretive* (see [EE87, Definition III.6.1]) if

(i) $W(T) \subset \Sigma_{\pi/2} = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$;

it is called *strictly accretive* if the inequality $\operatorname{Re} z \geq 0$ in (i) is replaced by the strict inequality $\operatorname{Re} z > 0$, and *uniformly accretive* if, instead of (i),

(i′) $W(T) \subset \alpha + \Sigma_{\pi/2} = \{z \in \mathbb{C} : \operatorname{Re} z \geq \alpha\}$ for some $\alpha > 0$.

The operator $T$ is called *regularly accretive* (see [Gom88]) if, instead of (i),

(i″) $W(T) \subset \Sigma_\omega$ for some $\omega \in [0, \pi/2)$,

it is called *regularly quasi-accretive* (or *regularly accretive with vertex* $\alpha$) if $T + \alpha$ is regularly accretive for some $\alpha \in \mathbb{R}$.

As in the bounded case, the numerical range $W(T)$ is convex (see [Sto32, Theorem 4.7]) and $\sigma_{\rm p}(T) \subset W(T)$. For the inclusion of the spectrum, an additional condition is required: If $T$ is closed, then $\sigma(T) \subset \overline{W(T)}$ holds if every component of $\mathbb{C} \setminus W(T)$ contains a point $\lambda_0 \in \rho(T)$ (see [Kat95, Section V.3.2]); then, as in the bounded case, the resolvent satisfies the norm estimate

$$\big\|(T-\lambda)^{-1}\big\| \leq \frac{1}{\operatorname{dist}\big(\lambda, W(T)\big)}, \quad \lambda \notin \overline{W(T)}.$$

**Remark 2.1.23** If $T$ is accretive and $(-\infty, 0) \cap \rho(T) \neq \emptyset$, then $T$ is m-accretive (see [Kat95, Section V.3.10]) and the above resolvent estimate coincides with (2.1.6). Using this additional property, we define strictly m-accretive, uniformly m-accretive, regularly m-accretive, and regularly quasi-m-accretive operators. One can show that $T$ is m-accretive if and only if it is maximal accretive, *i.e.* $T$ has no proper accretive extension (see [Phi59]). Obviously, $T$ is regularly m-accretive if and only if it is sectorial with angle $< \pi/2$; note that such operators are referred to as m-sectorial with vertex 0 by T. Kato (see [Kat95, Section V.3.10]).

## 2.2 Closedness and closability of block operator matrices

If the entries of a block operator matrix are densely defined and closable, this need not be true for the block operator matrix. Moreover, if a block operator matrix is closable, its closure need not have a block operator matrix representation anymore. In the following, we identify classes of closable block operator matrices and we describe their closures.

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Banach spaces. In the product space $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ we consider an unbounded block operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.2.1}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_1$, $B : \mathcal{H}_2 \to \mathcal{H}_1$, $C : \mathcal{H}_1 \to \mathcal{H}_2$, and $D : \mathcal{H}_2 \to \mathcal{H}_2$ are closable operators with dense domains $\mathcal{D}(A), \mathcal{D}(C) \subset \mathcal{H}_1$, $\mathcal{D}(B), \mathcal{D}(D) \subset \mathcal{H}_2$.

We always suppose that $\mathcal{A}$ with its natural domain

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}_1 \oplus \mathcal{D}_2 := \big(\mathcal{D}(A) \cap \mathcal{D}(C)\big) \oplus \big(\mathcal{D}(B) \cap \mathcal{D}(D)\big) \tag{2.2.2}$$

is also densely defined. Note that, unlike bounded operators, unbounded linear operators, in general, do not admit a matrix representation (2.2.1) with respect to a given decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

If either the entries $A$ and $D$ or the entries $B$ and $C$ are bounded, then $\mathcal{A}$ with its domain (2.2.2) is closable as the sum of a closable and a bounded operator; if, in addition, the unbounded entries are closed, then $\mathcal{A}$ is closed. We use the following classification of unbounded block operator matrices in terms of the positions of the dominating entries (see [Tre00], [Tre08]).

**Definition 2.2.1** The block operator matrix $\mathcal{A}$ in (2.2.1) is called

i) *diagonally dominant* if $C$ is $A$-bounded and $B$ is $D$-bounded,
ii) *off-diagonally dominant* if $A$ is $C$-bounded and $D$ is $B$-bounded,
iii) *upper dominant* if $C$ is $A$-bounded and $D$ is $B$-bounded,
iv) *lower dominant* if $A$ is $C$-bounded and $B$ is $D$-bounded.

The lower dominant case does not need extra consideration; it is equivalent to the upper dominant case if the two space components are exchanged.

In all cases above, the domain of the block operator matrix $\mathcal{A}$ is determined by the domains of the dominating entries,

$$\mathcal{D}(\mathcal{A}) = \begin{cases} \mathcal{D}(A) \oplus \mathcal{D}(D) & \text{if } \mathcal{A} \text{ is diagonally dominant,} \\ \mathcal{D}(C) \oplus \mathcal{D}(B) & \text{if } \mathcal{A} \text{ is off-diagonally dominant,} \\ \mathcal{D}(A) \oplus \mathcal{D}(B) & \text{if } \mathcal{A} \text{ is upper dominant;} \end{cases} \tag{2.2.3}$$

hence $\mathcal{A}$ is automatically densely defined since so are its entries.

If at least one of the entries in each column is closed, then the type of dominance may be read off from domain inclusions (see Remark 2.1.3 i)).

**Remark 2.2.2** The block operator matrix $\mathcal{A}$ is

i) diagonally dominant if $A, D$ are closed, $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$,
ii) off-diagonally dominant if $B, C$ are closed, $\mathcal{D}(C) \subset \mathcal{D}(A)$, $\mathcal{D}(B) \subset \mathcal{D}(D)$,
iii) upper dominant if $A, B$ are closed, $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(B) \subset \mathcal{D}(D)$.

In order to guarantee that $\mathcal{A}$ is closable or closed, we need more refined assumptions on the strength of the entries with respect to each other. First we consider the diagonally dominant and the off-diagonally dominant case (see [Tre08, Section 2]).

**Definition 2.2.3** Let $\delta \geq 0$. The block operator matrix $\mathcal{A}$ is called

i) *diagonally dominant of order* $\delta$ if $C$ is $A$-bounded with $A$-bound $\delta_C$, $B$ is $D$-bounded with $D$-bound $\delta_B$, and $\delta = \max\{\delta_B, \delta_C\}$,
ii) *off-diagonally dominant of order* $\delta$ if $A$ is $C$-bounded with $C$-bound $\delta_A$, $D$ is $B$-bounded with $B$-bound $\delta_D$, and $\delta = \max\{\delta_A, \delta_D\}$.

**Remark 2.2.4** If $\mathcal{H}_1 = \mathcal{H}_2$, then the block operator matrix $\mathcal{A}$ is off-diagonally dominant (of order $\delta$) if and only if

$$\mathcal{G}\mathcal{A} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix} \tag{2.2.4}$$

is diagonally dominant (of order $\delta$); in this particular situation, the following statements for the off-diagonally dominant case may be deduced from the corresponding statements for the diagonally dominant case.

**Proposition 2.2.5** *Define the block operator matrices*

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{2.2.5}$$

i) *If $\mathcal{A}$ is diagonally dominant of order $\delta$, then $\mathcal{S}$ is $\mathcal{T}$-bounded with $\mathcal{T}$-bound $\delta$.*
ii) *If $\mathcal{A}$ is off-diagonally dominant of order $\delta$, then $\mathcal{T}$ is $\mathcal{S}$-bounded with $\mathcal{S}$-bound $\delta$.*

***Proof.*** We prove i); the proof of ii) is similar. Let $\varepsilon > 0$ be arbitrary. By the assumptions and Remark 2.1.3 ii), there exist constants $a'_B$, $a'_C$, $b'_B$, $b'_C \geq 0$ such that $\delta_B \leq b'_B < \delta_B + \varepsilon$, $\delta_C \leq b'_C < \delta_C + \varepsilon$ and

$$\|Bg\|^2 \leq a'^2_B \|g\|^2 + b'^2_B \|Dg\|^2, \quad g \in \mathcal{D}(D), \tag{2.2.6}$$

$$\|Cf\|^2 \leq a'^2_C \|f\|^2 + b'^2_C \|Af\|^2, \quad f \in \mathcal{D}(A). \tag{2.2.7}$$

Hence we obtain, for $(f\ g)^{\mathrm{t}} \in \mathcal{D}(A) \oplus \mathcal{D}(D)$,

$$\begin{aligned}\left\| \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 &= \|Bg\|^2 + \|Cf\|^2 \\ &\leq a_B'^2 \|g\|^2 + b_B'^2 \|Dg\|^2 + a_C'^2 \|f\|^2 + b_C'^2 \|Af\|^2 \\ &\leq \bigl(\max\{a_B', a_C'\}\bigr)^2 \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 + \bigl(\max\{b_B', b_C'\}\bigr)^2 \left\| \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2.\end{aligned}$$

Since $\max\{b_B', b_C'\} < \max\{\delta_B + \varepsilon, \delta_C + \varepsilon\} = \delta + \varepsilon$, this shows that $\mathcal{S}$ is $\mathcal{T}$-bounded with $\mathcal{T}$-bound $< \delta$. That the $\mathcal{T}$-bound indeed equals $\delta$ follows by contradiction if we first set $f=0$ and then $g=0$ in the above inequality. □

An immediate consequence of Proposition 2.2.5 and Lemma 2.1.6 is:

**Corollary 2.2.6** *If, in Proposition* 2.2.5 i) *or* ii)*, the order of dominance* $\delta$ *is* $< 1$*, then* $\mathcal{S}$ *or* $\mathcal{T}$*, respectively, is* $\mathcal{A}$*-bounded with* $\mathcal{A}$*-bound* $\leq \delta/(1-\delta)$.

**Theorem 2.2.7** *The block operator matrix* $\mathcal{A}$ *is closable if*

i) $\mathcal{A}$ *is diagonally dominant of order* $< 1$ *with closure given by*

$$\overline{\mathcal{A}} = \begin{pmatrix} \overline{A} & B \\ C & \overline{D} \end{pmatrix};$$

*if, in addition,* $A$ *and* $D$ *are closed, then* $\mathcal{A}$ *is closed.*

ii) $\mathcal{A}$ *is off-diagonally dominant of order* $< 1$ *with closure given by*

$$\overline{\mathcal{A}} = \begin{pmatrix} A & \overline{B} \\ \overline{C} & D \end{pmatrix};$$

*if, in addition,* $B$ *and* $C$ *are closed, then* $\mathcal{A}$ *is closed.*

***Proof.*** All claims follow from Proposition 2.2.5, Theorem 2.1.4 i), and from the formulae

$$\overline{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}} = \begin{pmatrix} \overline{A} & 0 \\ 0 & \overline{D} \end{pmatrix}, \quad \overline{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}} = \begin{pmatrix} 0 & \overline{B} \\ \overline{C} & 0 \end{pmatrix}. \qquad \square$$

In the above theorem the assumptions are symmetric for the two columns of the block operator matrix. In fact, a stronger dominance in one column may compensate for a weaker one in the other column.

**Theorem 2.2.8** *The block operator matrix* $\mathcal{A}$ *is closable if*

i) $\mathcal{A}$ *is diagonally dominant and, for the relative bounds* $\delta_C$ *of* $C$*,* $\delta_B$ *of* $B$*,*

$$\delta_C^2\bigl(1+\delta_B^2\bigr) < 1 \quad \textit{or} \quad \delta_B^2\bigl(1+\delta_C^2\bigr) < 1;$$

*if, in addition,* $A$ *and* $D$ *are closed, then* $\mathcal{A}$ *is closed.*

ii) *$\mathcal{A}$ is off-diagonally dominant and, for the relative bounds $\delta_A$ of $A$ and $\delta_D$ of $D$,*

$$\delta_A^2\big(1+\delta_D^2\big) < 1 \quad \text{or} \quad \delta_D^2\big(1+\delta_A^2\big) < 1;$$

*if, in addition, $B$ and $C$ are closed, then $\mathcal{A}$ is closed.*

***Proof.*** We prove i); the proof of ii) is analogous. Let *e.g.* $\delta_C^2(1+\delta_B^2) < 1$. Then $\delta_C < 1$ and $\delta_B\delta_C < 1$. We consider the block operator matrices

$$\mathcal{T} := \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

First we prove that $\mathcal{T}$ is closable. Suppose that $\big((x_n\ y_n)^{\mathrm{t}}\big)_1^\infty \subset \mathcal{D}(A)\oplus\mathcal{D}(D)$ is a sequence such that $(x_n\ y_n)^{\mathrm{t}} \to (0\ 0)^{\mathrm{t}}$, $n\to\infty$, and

$$\mathcal{T}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} Ax_n + By_n \\ Dy_n \end{pmatrix} \longrightarrow \begin{pmatrix} v \\ w \end{pmatrix}, \quad n\to\infty,$$

with some $v\in\mathcal{H}_1$, $w\in\mathcal{H}_2$. Since $D$ is closable, this shows that $w=0$. The assumption that $B$ is $D$-bounded implies that there exist $a_B, b_B \geq 0$ with

$$\|By_n\| \leq a_B\|y_n\| + b_B\|Dy_n\| \to 0, \quad n\to\infty.$$

Hence $By_n \to 0$ and $Ax_n \to v$, $n\to\infty$. Then $v=0$ because $A$ is closable, which completes the proof that $\mathcal{T}$ is closable. In a similar way, one can show that $\mathcal{T}$ is closed if $A$ and $D$ are closed.

By Theorem 2.1.4 i), it suffices to prove that $\mathcal{S}$ is $\mathcal{T}$-bounded with $\mathcal{T}$-bound $< 1$. Choose $\varepsilon > 0$ so that $(\delta_C+\varepsilon)^2(1+(\delta_B+\varepsilon)^2) < 1$ and let the constants $a'_B$, $a'_C$, $b'_B$, $b'_C \geq 0$ with $\delta_B \leq b'_B < \delta_B+\varepsilon$, $\delta_C \leq b'_C < \delta_C+\varepsilon$ be as in (2.2.6), (2.2.7); in particular, we have $b'^2_C\big(1+b'^2_B\big) < 1$. For $(f\ g)^{\mathrm{t}} \in \mathcal{D}(A)\oplus\mathcal{D}(D)$ and arbitrary $\gamma > 0$, we obtain, using the inequality (2.1.3),

$$\begin{aligned}
&\left\|\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}\right\|^2 = \|Cf\|^2 \\
&\leq a'^2_C\|f\|^2 + b'^2_C\big(\|Af+Bg\|+\|Bg\|\big)^2 \\
&\leq a'^2_C\|f\|^2 + (1+\gamma^{-1})b'^2_C\|Af+Bg\|^2 + (1+\gamma)b'^2_C\|Bg\|^2 \\
&\leq a'^2_C\|f\|^2 + (1+\gamma)a'^2_B b'^2_C\|g\|^2 + (1+\gamma^{-1})b'^2_C\|Af+Bg\|^2 + (1+\gamma)b'^2_B b'^2_C\|Dg\|^2 \\
&\leq \max\big\{a'^2_C, (1+\gamma)a'^2_B b'^2_C\big\}\left\|\begin{pmatrix} f \\ g \end{pmatrix}\right\|^2 + b'^2_C\max\big\{1+\gamma^{-1}, (1+\gamma)b'^2_B\big\}\left\|\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}\right\|^2.
\end{aligned}$$

There exists a $\gamma > 0$ with $b'^2_C\max\big\{1+\gamma^{-1}, (1+\gamma)b'^2_B\big\} < 1$ if and only if

$$\frac{b'^2_C}{1-b'^2_C} < \frac{1-b'^2_B b'^2_C}{b'^2_B b'^2_C};$$

the latter is satisfied as the equivalent condition $b'^2_C\big(1+b'^2_B\big) < 1$ holds. □

**Corollary 2.2.9** *The block operator matrix $\mathcal{A}$ is closable if*

i) *$C$ is $A$-bounded, $B$ is $D$-bounded and at least one of them has relative bound $0$; if, in addition, $A$ and $D$ are closed, then $\mathcal{A}$ is closed.*
ii) *$A$ is $C$-bounded, $D$ is $B$-bounded and at least one of them has relative bound $0$; if, in addition, $B$ and $C$ are closed, then $\mathcal{A}$ is closed.*

**Remark 2.2.10** The assumptions of Corollary 2.2.9 are satisfied if either $\mathcal{A}$ is diagonally dominant and one of the off-diagonal elements is bounded or if $\mathcal{A}$ is off-diagonally dominant and one of the diagonal elements is bounded.

Proposition 2.1.19 and Corollary 2.1.20 together with Corollary 2.2.9 yield some useful criteria for a block operator matrix to be closable or closed; here we restrict ourselves to closedness and to the Hilbert space case.

**Corollary 2.2.11** *The block operator matrix $\mathcal{A}$ is closed if $A$ and $D$ are closed and one of the following holds:*

i) $\mathcal{D}(A) \subset \mathcal{D}(C)$ *and* $\mathcal{D}(|D|^\gamma) \subset \mathcal{D}(B)$ *for some* $\gamma \in (0,1)$,
i′) $\mathcal{D}(D) \subset \mathcal{D}(B)$ *and* $\mathcal{D}(|A|^\gamma) \subset \mathcal{D}(C)$ *for some* $\gamma \in (0,1)$,

*or if $B$ and $C$ are closed and one of the following holds:*

ii) $\mathcal{D}(C) \subset \mathcal{D}(A)$ *and* $\mathcal{D}(|B|^\gamma) \subset \mathcal{D}(D)$ *for some* $\gamma \in (0,1)$,
ii′) $\mathcal{D}(B) \subset \mathcal{D}(D)$ *and* $\mathcal{D}(|C|^\gamma) \subset \mathcal{D}(A)$ *for some* $\gamma \in (0,1)$.

***Proof.*** If i) holds, then $C$ is $A$-bounded and $B$ is $D$-bounded with $D$-bound 0 by Corollary 2.1.20. Hence $\mathcal{A}$ is closed by Corollary 2.2.9 i). The proof for the other cases is analogous. □

Under weaker assumptions than in the previous theorems, the closure of a block operator matrix need not be a block operator matrix. This means that, in general, the domain of the closure is non-diagonal (see [Nag90], [Nag89]), *i.e.* it does not have the form $\widetilde{\mathcal{D}}_1 \oplus \widetilde{\mathcal{D}}_2$ with $\widetilde{\mathcal{D}}_1 \subset \mathcal{H}_1$, $\widetilde{\mathcal{D}}_2 \subset \mathcal{H}_2$.

In the following, we characterize the closablility of block operator matrices and describe their closures in terms of the Schur complements.

**Definition 2.2.12** The operator functions $S_1$ and $S_2$ defined by

$$S_1(\lambda) := A - \lambda - B(D-\lambda)^{-1}C, \quad \lambda \in \rho(D), \tag{2.2.8}$$

$$S_2(\lambda) := D - \lambda - C(A-\lambda)^{-1}B, \quad \lambda \in \rho(A), \tag{2.2.9}$$

are called *Schur complements* of $\mathcal{A}$.

The values $S_1(\lambda)$ and $S_2(\lambda)$ are linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. In contrast to the bounded case (see Definition 1.6.1), they need not be bounded and their domains $\mathcal{D}\big(S_1(\lambda)\big)$ and $\mathcal{D}\big(S_2(\lambda)\big)$ may vary with $\lambda$.

**Remark 2.2.13** We have

$$\mathcal{D}\big(S_1(\lambda)\big) = \mathcal{D}(A) \cap \mathcal{D}(C) = \mathcal{D}_1 \qquad \text{if } \mathcal{D}(D) \subset \mathcal{D}(B),$$
$$\mathcal{D}\big(S_2(\lambda)\big) = \mathcal{D}(B) \cap \mathcal{D}(D) = \mathcal{D}_2 \qquad \text{if } \mathcal{D}(A) \subset \mathcal{D}(C),$$

independently of $\lambda \in \rho(D)$ and $\lambda \in \rho(A)$, respectively; in particular,

$$\mathcal{D}\big(S_1(\lambda)\big) = \begin{cases} \mathcal{D}(A) & \text{if } \mathcal{A} \text{ is diagonally dominant,} \\ \mathcal{D}(C) & \text{if } \mathcal{A} \text{ is lower dominant,} \end{cases}$$

$$\mathcal{D}\big(S_2(\lambda)\big) = \begin{cases} \mathcal{D}(D) & \text{if } \mathcal{A} \text{ is diagonally dominant,} \\ \mathcal{D}(B) & \text{if } \mathcal{A} \text{ is upper dominant.} \end{cases}$$

The next theorem was proved in [ALMS94, Theorem 1.1, Proposition 3.1] for upper dominant block operator matrices; in the more general form presented here, it was established in [Shk95, Theorem 1.1].

The *Frobenius-Schur factorizations* (2.2.10), (2.2.12) below generalize the corresponding formulae in the bounded case (see (1.6.2), (1.6.3)); later we also use them to relate the spectrum of the block operator matrix to the spectra of its Schur complements.

**Theorem 2.2.14** *Suppose that $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\rho(A) \neq \emptyset$, and that, for some (and hence for all) $\mu \in \rho(A)$, the operator $(A-\mu)^{-1}B$ is bounded on $\mathcal{D}(B)$. Then $\mathcal{A}$ is closable (closed, respectively) if and only if $S_2(\mu)$ is closable (closed, respectively) for some (and hence for all) $\mu \in \rho(A)$; in this case, the closure $\overline{\mathcal{A}}$ is given by*

$$\overline{\mathcal{A}} = \mu + \begin{pmatrix} I & 0 \\ C(A-\mu)^{-1} & I \end{pmatrix} \begin{pmatrix} A-\mu & 0 \\ 0 & \overline{S_2(\mu)} \end{pmatrix} \begin{pmatrix} I & \overline{(A-\mu)^{-1}B} \\ 0 & I \end{pmatrix}, \tag{2.2.10}$$

*independently of $\mu \in \rho(A)$, that is,*

$$\mathcal{D}(\overline{\mathcal{A}}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 : \, x + \overline{(A-\mu)^{-1}B}\, y \in \mathcal{D}(A),\, y \in \mathcal{D}\big(\overline{S_2(\mu)}\big) \right\},$$

$$\overline{\mathcal{A}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A-\mu)\big(x + \overline{(A-\mu)^{-1}B}\, y\big) + \mu\, x \\ C\big(x + \overline{(A-\mu)^{-1}B}\, y\big) + \big(\overline{S_2(\mu)} + \mu\big)\, y \end{pmatrix}.$$

***Proof.*** First we note that the assumptions on $(A-\mu)^{-1}B$ and on $S_2(\mu)$ do not depend on the choice of $\mu \in \rho(A)$. In fact, for $\mu, \mu_0 \in \rho(A)$, the differences $(A-\mu_0)^{-1}B - (A-\mu)^{-1}B$ and $S_2(\mu_0) - S_2(\mu)$, respectively, are bounded since, by the resolvent identity for $A$,

$$(A-\mu_0)^{-1}B - (A-\mu)^{-1}B = (\mu_0 - \mu)(A-\mu_0)^{-1}(A-\mu)^{-1}B,$$
$$S_2(\mu_0) - S_2(\mu) = -(\mu_0 - \mu)\big(I + C(A-\mu_0)^{-1}(A-\mu)^{-1}B\big).$$

By direct computation, we see that

$$\mathcal{A}-\mu=\begin{pmatrix} I & 0\\ C(A-\mu)^{-1} & I\end{pmatrix}\begin{pmatrix} A-\mu & 0\\ 0 & S_2(\mu)\end{pmatrix}\begin{pmatrix} I & (A-\mu)^{-1}B\\ 0 & I\end{pmatrix}. \quad (2.2.11)$$

Note that in (2.2.11) we can replace $(A-\mu)^{-1}B = \overline{(A-\mu)^{-1}B}\,\big|_{\mathcal{D}(B)}$ by $\overline{(A-\mu)^{-1}B}$ since the domain of the middle factor is equal to

$$\mathcal{D}(A)\oplus\mathcal{D}\big(S_2(\mu)\big)=\mathcal{D}(A)\oplus\big(\mathcal{D}(B)\cap\mathcal{D}(D)\big)\subset\mathcal{D}(A)\oplus\mathcal{D}(B).$$

Then the first and last factor in (2.2.11) are bounded and boundedly invertible in $\mathcal{H}_1\oplus\mathcal{H}_2$. Hence $\mathcal{A}-\mu$ is closable (closed, respectively) if and only if so is the middle factor in (2.2.11); since $\rho(A)\neq\emptyset$ implies that $A$ is closed, the latter holds if and only if $S_2(\mu)$ is closable (closed, respectively). In this case, the closure of the product of the three operator matrices is obtained by taking the closure of the middle factor. This proves (2.2.10). Since $\overline{\mathcal{A}}$ does not depend on $\mu$, the right hand side of (2.2.10) does not either. □

**Remark 2.2.15** If $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, then $(A-\mu)^{-1}B$ is bounded on $\mathcal{D}(B)$ if and only if $\mathcal{D}(A^*)\subset\mathcal{D}(B^*)$.

***Proof.*** We follow the proof of [ALMS94, Proposition 3.1]. If $\mathcal{D}(A^*)\subset\mathcal{D}(B^*)$, then $(A-\mu)^{-1}B\subset\big(B^*(A^*-\overline{\mu})^{-1}\big)^*$ is bounded by the closed graph theorem. So the densely defined operator $(A-\mu)^{-1}B$ is closable and $\overline{(A-\mu)^{-1}B}=\big(B^*(A^*-\overline{\mu})^{-1}\big)^*$. Conversely, if $(A-\mu)^{-1}B$ is bounded on the dense set $\mathcal{D}(B)$, then $\overline{(A-\mu)^{-1}B}$ is everywhere defined and bounded; hence so is $\big(B^*(A^*-\overline{\mu})^{-1}\big)^*\supset\overline{(A-\mu)^{-1}B}$ which yields $\mathcal{D}(A^*)\subset\mathcal{D}(B^*)$. □

An example for the subtle difference between closed and closable block operator matrices was given in [ADFG00, Section 6] where accretive block operator matrices were considered. Block operator matrices of this form also occur in stability problems in hydrodynamics; there information about the closure is needed so that results for contractive semigroups can be applied (see [AHKM03, Remark 3.12, Theorem 3.15]).

**Example 2.2.16** Suppose that $A$ and $D$ are uniformly positive self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $A\geq\alpha>0$ and $D\geq\delta>0$, and $B$ is a densely defined closable operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. If $A$ and $B$ are unbounded with $\mathcal{D}(A^{1/2})\subset\mathcal{D}(B^*)$ and $D$ is bounded, then

$$\mathcal{A}=\begin{pmatrix} A & B\\ -B^* & D\end{pmatrix}$$

is closable, but not closed; this follows from Theorem 2.2.14 and from Remark 2.2.15 since $B^*A^{-1}B$ is bounded on the dense set $\mathcal{D}(B)$ so that

$$S_2(0) = D - B^*A^{-1}B \subsetneq D - \overline{B^*A^{-1}B} = \overline{S_2(0)}$$

is bounded on $\mathcal{D}(B)$ and thus closable, but not closed. If, however, $D$ is unbounded and $\mathcal{D}(D)\subset\mathcal{D}(B)$, then $\mathcal{A}$ is closed by Corollary 2.2.11 i′). Since

$$\operatorname{Re}(\mathcal{A}\mathbf{x},\mathbf{x}) = \operatorname{Re}(Ax,x) + \operatorname{Re}(Dy,y) \geq \max\{\alpha,\delta\}\|\mathbf{x}\|^2$$

for $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{D}(A) \oplus \big(\mathcal{D}(B)\cap\mathcal{D}(D)\big)$, $\mathcal{A}$ is uniformly m-accretive if $D$ is unbounded and $\mathcal{D}(D) \subset \mathcal{D}(B)$, but not if $D$ is bounded; in this case, only the closure $\overline{\mathcal{A}}$, which is given by (2.2.10), is uniformly m-accretive.

**Remark 2.2.17** General criteria for unbounded block operator matrices to be accretive and regularly accretive were derived in [Arl02] (there, like in [Kat95], the notion sectorial is used for regularly accretive). In addition, m-accretive and regularly m-accretive extensions are parametrized in terms of the Schur complements and their derivatives.

The following analogue of Theorem 2.2.14 in terms of the first Schur complement is proved in a completely analogous way.

**Theorem 2.2.18** *Suppose that $\mathcal{D}(D) \subset \mathcal{D}(B)$, $\rho(D) \neq \emptyset$, and that, for some (and hence for all) $\mu \in \rho(D)$, the operator $(D-\mu)^{-1}C$ is bounded on $\mathcal{D}(C)$. Then $\mathcal{A}$ is closable (closed, respectively) if and only if $S_1(\mu)$ is closable (closed, respectively) for some (and hence for all) $\mu \in \rho(D)$; in this case, the closure $\overline{\mathcal{A}}$ is given by*

$$\overline{\mathcal{A}} = \mu + \begin{pmatrix} I & B(D-\mu)^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} \overline{S_1(\mu)} & 0 \\ 0 & D-\mu \end{pmatrix}\begin{pmatrix} I & 0 \\ \overline{(D-\mu)^{-1}C} & I \end{pmatrix}, \quad (2.2.12)$$

*independently of $\mu \in \rho(D)$, that is,*

$$\mathcal{D}(\overline{\mathcal{A}}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 : \ x \in \mathcal{D}\big(\overline{S_1(\mu)}\big),\ \overline{(D-\mu)^{-1}C}\,x + y \in \mathcal{D}(D) \right\},$$

$$\overline{\mathcal{A}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \big(\overline{S_1(\mu)} + \mu\big)\,x + B\big(\overline{(D-\mu)^{-1}C}\,x + y\big) \\ (D-\mu)\big(\overline{(D-\mu)^{-1}C}\,x + y\big) + \mu\, y \end{pmatrix}.$$

**Remark 2.2.19** If $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, then $(D-\mu)^{-1}C$ is bounded on $\mathcal{D}(C)$ if and only if $\mathcal{D}(D^*) \subset \mathcal{D}(C^*)$.

Apart from the Schur complements, there is another pair of operator functions that reflect the spectral properties of a block operator matrix. These so-called quadratic complements may be used, in particular, if neither $\mathcal{D}(A) \subset \mathcal{D}(C)$ nor $\mathcal{D}(D) \subset \mathcal{D}(B)$; in this case, the domains of the Schur complements are, in general, not independent of $\lambda$.

**Definition 2.2.20** Suppose that either $C$ or $B$ is boundedly invertible. Then the quadratic operator polynomials $T_1$ and $T_2$ defined by

$$T_1(\lambda) := C - (D-\lambda)B^{-1}(A-\lambda) \quad \text{if } B \text{ is boundedly invertible,} \quad (2.2.13)$$

$$T_2(\lambda) := B - (A-\lambda)C^{-1}(D-\lambda) \quad \text{if } C \text{ is boundedly invertible,} \quad (2.2.14)$$

for $\lambda \in \mathbb{C}$ are called *quadratic complements* of $\mathcal{A}$.

The values $T_1(\lambda)$ and $T_2(\lambda)$ are linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and from $\mathcal{H}_2$ to $\mathcal{H}_1$, respectively; their domains $\mathcal{D}\big(T_1(\lambda)\big)$ and $\mathcal{D}\big(T_2(\lambda)\big)$ may, in general, depend on $\lambda$.

**Remark 2.2.21** We have

$$\begin{aligned} \mathcal{D}\big(T_1(\lambda)\big) &= \mathcal{D}(A) \cap \mathcal{D}(C) = \mathcal{D}_1 \qquad \text{if } \mathcal{D}(B) \subset \mathcal{D}(D), \\ \mathcal{D}\big(T_2(\lambda)\big) &= \mathcal{D}(B) \cap \mathcal{D}(D) = \mathcal{D}_2 \qquad \text{if } \mathcal{D}(C) \subset \mathcal{D}(A), \end{aligned}$$

independently of $\lambda \in \mathbb{C}$; in particular, if $\mathcal{A}$ is off-diagonally dominant,

$$\mathcal{D}\big(T_1(\lambda)\big) = \mathcal{D}(C), \quad \mathcal{D}\big(T_2(\lambda)\big) = \mathcal{D}(B).$$

In the special case $\mathcal{H}_1 = \mathcal{H}_2$, there is a certain relation between quadratic complements and Schur complements.

**Remark 2.2.22** If $\mathcal{H}_1 = \mathcal{H}_2$, then, with $\mathcal{G}$ as in (2.2.4), we have

$$\mathcal{G}(\mathcal{A}-\lambda) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix} = \begin{pmatrix} C & D-\lambda \\ A-\lambda & B \end{pmatrix} =: \mathcal{A}_\lambda.$$

Thus, the quadratic complements $T_1(\lambda)$, $T_2(\lambda)$ may be identified with the first and second, respectively, Schur complement of $\mathcal{A}_\lambda$ evaluated at 0; in this case, the following results using the quadratic complements may be deduced from the corresponding results using the Schur complements.

**Theorem 2.2.23** *Suppose that $\mathcal{D}(C) \subset \mathcal{D}(A)$, $C$ is boundedly invertible, and that $C^{-1}D$ is bounded on $\mathcal{D}(D)$. Then $\mathcal{A}$ is closable* (*closed, respectively*) *if and only if $T_2(\mu)$ is closable* (*closed, respectively*) *for some* (*and hence for all*) *$\mu \in \mathbb{C}$; in this case, the closure $\overline{\mathcal{A}}$ is given by*

$$\overline{\mathcal{A}} = \mu + \begin{pmatrix} I & (A-\mu)C^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & \overline{T_2(\mu)} \\ C & 0 \end{pmatrix} \begin{pmatrix} I & \overline{C^{-1}(D-\mu)} \\ 0 & I \end{pmatrix}, \qquad (2.2.15)$$

*independently of $\mu \in \mathbb{C}$, that is,*

$$\mathcal{D}(\overline{\mathcal{A}}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 : \, x + \overline{C^{-1}(D-\mu)}\, y \in \mathcal{D}(C),\, y \in \mathcal{D}\big(\overline{T_2(\mu)}\big) \right\},$$

$$\overline{\mathcal{A}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A-\mu)\big(x + \overline{C^{-1}(D-\mu)}\, y\big) + \mu\, x + \overline{T_2(\mu)}\, y \\ C\big(x + \overline{C^{-1}(D-\mu)}\, y\big) + \mu\, y \end{pmatrix}.$$

***Proof.*** Obviously, for every $\mu \in \mathbb{C}$,

$$T_2(\mu) = B - AC^{-1}D + \mu\big(AC^{-1} + C^{-1}D\big) - \mu^2 C^{-1},$$

the coefficients of $\mu$ and $\mu^2$ being bounded operators due to the assumptions. Hence the closability of $T_2(\mu)$ does not depend on the choice of $\mu \in \mathbb{C}$. By direct computation, we find that

$$\mathcal{A} - \mu = \begin{pmatrix} I & (A-\mu)C^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & T_2(\mu) \\ C & 0 \end{pmatrix} \begin{pmatrix} I & C^{-1}(D-\mu) \\ 0 & I \end{pmatrix}. \tag{2.2.16}$$

Note that in (2.2.16) we can replace $C^{-1}(D-\mu) = \overline{C^{-1}(D-\mu)}\,\big|_{\mathcal{D}(D)}$ by $\overline{C^{-1}(D-\mu)}$ since the domain of the middle factor is equal to

$$\mathcal{D}(C) \oplus \mathcal{D}\big(T_2(\mu)\big) = \mathcal{D}(C) \oplus \big(\mathcal{D}(B) \cap \mathcal{D}(D)\big) \subset \mathcal{D}(C) \oplus \mathcal{D}(D).$$

Then the first and last factor in (2.2.16) are bounded and boundedly invertible. Hence $\mathcal{A}-\mu$ is closable if and only if so is the middle factor in (2.2.16); since $C$ is boundedly invertible and thus closed, the latter holds if and only if $T_2(\mu)$ is closable. In this case, the closure is obtained by taking the closure of the middle factor. This proves (2.2.15). Since $\overline{\mathcal{A}}$ does not depend on $\mu$, the same is true for the right hand side of (2.2.15). $\square$

**Remark 2.2.24** If $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, then $C^{-1}(D-\mu)$ is bounded on $\mathcal{D}(D)$ if and only if $\mathcal{D}(C^*) \subset \mathcal{D}(D^*)$.

If $B$ is boundedly invertible, the following analogue of Theorem 2.2.23 can be proved.

**Theorem 2.2.25** *Suppose that $\mathcal{D}(B) \subset \mathcal{D}(D)$, $B$ is boundedly invertible, and that $B^{-1}A$ is bounded on $\mathcal{D}(A)$. Then $\mathcal{A}$ is closable* (*closed, respectively*) *if and only if $T_1(\mu)$ is closable* (*closed, respectively*) *for some* (*and hence for all*) *$\mu \in \mathbb{C}$; in this case, the closure $\overline{\mathcal{A}}$ is given by*

$$\overline{\mathcal{A}} = \mu + \begin{pmatrix} I & 0 \\ (D-\mu)B^{-1} & I \end{pmatrix} \begin{pmatrix} 0 & B \\ \overline{T_1(\mu)} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \overline{B^{-1}(A-\mu)} & I \end{pmatrix}, \tag{2.2.17}$$

*independently of $\mu \in \mathbb{C}$, that is,*

$$\mathcal{D}(\overline{\mathcal{A}}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2 : \, x \in \mathcal{D}\big(\overline{T_1(\mu)}\big),\, \overline{B^{-1}(A-\mu)}\,x + y \in \mathcal{D}(B) \right\},$$

$$\overline{\mathcal{A}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B\big(\overline{B^{-1}(A-\mu)}\,x + y\big) + \mu\, x \\ (D-\mu)\big(\overline{B^{-1}(A-\mu)}\,x + y\big) + \overline{T_1(\mu)}\,x + \mu\, y \end{pmatrix}.$$

**Remark 2.2.26** If $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, then $B^{-1}(A-\mu)$ is bounded on $\mathcal{D}(A)$ if and only if $\mathcal{D}(B^*) \subset \mathcal{D}(A^*)$.

In the diagonally dominant and off-diagonally dominant case, the closedness of the Schur complements and of the quadratic complements can be related to the closedness criteria given in Theorems 2.2.7 and 2.2.8. Here $a_A, a_B, a_C, a_D$ and $b_A, b_B, b_C, b_D$ denote the constants in the inequalities for the respective relative boundedness properties (see (2.1.1)).

**Corollary 2.2.27** *The block operator matrix $\mathcal{A}$ is closed if one of the following assumptions holds:*

i) *$\mathcal{A}$ is diagonally dominant, $\rho(A) \neq \emptyset$, $D$ is closed, $(A-\mu)^{-1}B$ is bounded on $\mathcal{D}(B)$ for some $\mu \in \rho(A)$, and*

$$\big((a_C + |\mu|\, b_C)\|(A-\mu)^{-1}\| + b_C\big)b_B < 1; \tag{2.2.18}$$

ii) *$\mathcal{A}$ is diagonally dominant, $\rho(D) \neq \emptyset$, $A$ is closed, $(D-\mu)^{-1}C$ is bounded on $\mathcal{D}(C)$ for some $\mu \in \rho(D)$, and*

$$\big((a_B + |\mu|\, b_B)\|(D-\mu)^{-1}\| + b_B\big)b_C < 1; \tag{2.2.19}$$

iii) *$\mathcal{A}$ is off-diagonally dominant, $C$ is boundedly invertible, $B$ is closed, $C^{-1}D$ is bounded on $\mathcal{D}(D)$, and*

$$\big(a_A\|C^{-1}\| + b_A\big)b_D < 1;$$

iv) *$\mathcal{A}$ is off-diagonally dominant, $B$ is boundedly invertible, $C$ is closed, $B^{-1}A$ is bounded on $\mathcal{D}(A)$, and*

$$\big(a_D\|B^{-1}\| + b_D\big)b_A < 1.$$

***Proof.*** i) In Theorem 2.2.14 the diagonally dominant operator $\mathcal{A}$ is closed if $(A-\mu)^{-1}B$ is bounded on $\mathcal{D}(B)$ and $S_2(\mu)$ is closed for some $\mu \in \rho(A)$. Since $C$ is $A$-bounded and $B$ is $D$-bounded, there are $a_C, b_C, a_B, b_B \geq 0$ with

$$\begin{aligned}\|C(A-\mu)^{-1}Bx\| &\leq (a_C + |\mu|\, b_C)\|(A-\mu)^{-1}Bx\| + b_C\|Bx\| \\ &\leq \big((a_C + |\mu|\, b_C)\|(A-\mu)^{-1}\| + b_C\big)\big(b_B\|x\| + b_B\|Dx\|\big)\end{aligned}$$

for $x \in \mathcal{D}(D)$. Now $S_2(\mu) = D - \mu - C(A-\mu)^{-1}B$ is closed if $C(A-\mu)^{-1}B$ is $D$-bounded with $D$-bound $< 1$, which is satisfied if (2.2.18) holds.

The proof of ii) and for the off-diagonally dominant case is similar. □

**Remark 2.2.28** The above criteria compare to those derived in Theorems 2.2.7 and 2.2.8 as follows; we restrict ourselves to the diagonally dominant case and let $\delta_C$ denote the $A$-bound of $C$ and $\delta_B$ the $D$-bound of $B$.

i) If $\max\{\delta_C, \delta_B\} < 1$, then Theorem 2.2.7 i) shows that $\mathcal{A}$ is closed without any further assumptions.

ii) The weaker condition $\delta_B\delta_C < 1$ is necessary for Corollary 2.2.27 i) or ii) as well as for Theorem 2.2.8 i); in the latter, the sufficient conditions $\delta_C^2\big(1+\delta_B^2\big) < 1$ or $\delta_B^2\big(1+\delta_C^2\big) < 1$ do not require any assumptions linking $A$ to $B$ or $C$ to $D$.

The results of Theorems 2.2.14, 2.2.18, 2.2.23, and 2.2.25 should also be compared with the following results established in [Eng96, Proposition 2.2, Theorem 2.15] in view of applications to systems of evolution equations. Under the assumptions therein, more direct factorizations than in the previous theorems can be used. However, in general, not the closure but only a closed extension of the block operator matrix $\mathcal{A}$ is obtained.

**Remark 2.2.29** The block operator matrix $\mathcal{A}$ is closable if:

i) $\mathcal{A}$ is diagonally dominant, $A$, $D$ are boundedly invertible, $A^{-1}B$, $D^{-1}C$ are bounded on $\mathcal{D}(B)$ and $\mathcal{D}(C)$, respectively; a closed extension of $\mathcal{A}$ is

$$\mathcal{D}\big(\widetilde{\mathcal{A}}\big)=\left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathcal{H}_1\oplus\mathcal{H}_2 : x+\overline{A^{-1}B}y\in\mathcal{D}(A), \overline{D^{-1}C}x+y\in\mathcal{D}(D)\right\},$$
$$\widetilde{\mathcal{A}}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}A\big(x+\overline{A^{-1}B}y\big)\\ D\big(\overline{D^{-1}C}x+y\big)\end{pmatrix},$$

ii) $\mathcal{A}$ is off-diagonally dominant, $B, C$ are boundedly invertible, $B^{-1}A$, $C^{-1}D$ are bounded on $\mathcal{D}(A)$ and $\mathcal{D}(D)$, respectively; a closed extension of $\mathcal{A}$ is

$$\mathcal{D}\big(\widetilde{\mathcal{A}}\big)=\left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathcal{H}_1\oplus\mathcal{H}_2 : \overline{B^{-1}A}x+y\in\mathcal{D}(B), x+\overline{C^{-1}D}y\in\mathcal{D}(C)\right\},$$
$$\widetilde{\mathcal{A}}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}B\big(\overline{B^{-1}A}x+y\big)\\ C\big(x+\overline{C^{-1}D}y\big)\end{pmatrix}.$$

***Proof.*** i) Since $A$ and $D$ are boundedly invertible, we can write

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}I & A^{-1}B\\ D^{-1}C & I\end{pmatrix}. \tag{2.2.20}$$

By assumption, the first factor is closed and $A^{-1}B$, $D^{-1}C$ have bounded everywhere defined closures. Thus

$$\widetilde{\mathcal{A}}:=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}I & \overline{A^{-1}B}\\ \overline{D^{-1}C} & I\end{pmatrix},$$

is a closed extension of the right hand side of (2.2.20) and hence of $\mathcal{A}$. The proof of ii) is analogous. □

Remark 2.2.29 is a special case of [Eng96, Proposition 2.2, Theorem 2.15]; for i) we choose $X=[\mathcal{D}(A)]$, $Y=[\mathcal{D}(D)]$, $K=\overline{A^{-1}B}$, $L=\overline{D^{-1}C}$ therein, for ii) we choose $X=[\mathcal{D}(C)]$, $Y=[\mathcal{D}(B)]$, $G=\overline{C^{-1}D}$, $H=\overline{B^{-1}A}$.

Note that, in general, Remark 2.2.29 only yields a closed extension $\widetilde{\mathcal{A}}$ of $\mathcal{A}$, not the closure as in Theorems 2.2.14, 2.2.18, 2.2.23, and 2.2.25. This is due to the following difference between the factorization (2.2.20) and the factorizations (2.2.10), (2.2.15), respectively: In the latter, both outer factors are bounded and boundedly invertible, whereas in the former the first factor is boundedly invertible, but not bounded; so, in general, the closure of the right hand side cannot be obtained simply by taking the closure of the second factor.

## 2.3 Spectrum and resolvent

In this section we relate the spectrum of an unbounded block operator matrix $\mathcal{A}$ to the spectra of its Schur complements and of its quadratic complements. Furthermore, we give representations of the resolvent in terms of the Schur complements and of the quadratic complements. These relations are based on the factorizations for $\mathcal{A}-\lambda$ established in the previous section.

First we need the definitions of the various parts of the spectrum for operator functions (compare Section 1.6 for the bounded case) and a lemma on the spectra of equivalent linear operators and operator functions.

**Definition 2.3.1** Let $E, F$ be Banach spaces, let $\Omega\subset\mathbb{C}$ be open, and let $S$ be an operator function on $\Omega$ whose values $S(\lambda)$ are closed linear operators from $E$ to $F$. We define the *resolvent set* and *spectrum* of $S$ by

$$\begin{aligned}\rho(S) &:= \big\{\lambda\in\Omega : 0\in\rho\big(S(\lambda)\big)\big\},\\ \sigma(S) &:= \big\{\lambda\in\Omega : 0\in\sigma\big(S(\lambda)\big)\big\} = \Omega\setminus\rho(S);\end{aligned}$$

analogously, we define the *point spectrum*, *continuous spectrum*, and *residual spectrum* of $S$. The *numerical range* of $S$ is defined as

$$W(S) := \big\{\lambda\in\Omega : 0\in W\big(S(\lambda)\big)\big\}.$$

Note that for the particular case $S(\lambda)=T-\lambda$, $\lambda\in\mathbb{C}$, with a closed linear operator $T$ the above definitions coincide with those for the operator $T$ (see Definition 2.1.1 and (2.1.8)).

**Lemma 2.3.2** *Let $E, F, G$, and $H$ be Banach spaces, let $\Omega\subset\mathbb{C}$ be open, and let $S$, $T$ be operator functions on $\Omega$ whose values $S(\lambda)$, $T(\lambda)$ are closed linear operators from $E$ to $F$ and from $G$ to $H$, respectively. If $S$ and $T$*

*are equivalent on* $\Omega$, *i.e. for every* $\lambda \in \Omega$ *there exist bounded and boundedly invertible operators* $W_1(\lambda) : H \to F$ *and* $W_2(\lambda) : E \to G$ *so that*

$$T(\lambda) = W_1(\lambda) S(\lambda) W_2(\lambda), \quad \lambda \in \Omega, \tag{2.3.1}$$

*then*

$$\sigma(T) = \sigma(S), \quad \sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(S), \quad \sigma_{\mathrm{c}}(T) = \sigma_{\mathrm{c}}(S), \quad \sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{r}}(S).$$

***Proof.*** According to the assumptions on $W_1$ and $W_2$, it is sufficient to prove the inclusions in one direction. First we note that (2.3.1) is an operator equality; in particular, $\mathcal{D}\big(T(\lambda)\big) = \big\{x \in E : W_2(\lambda) \in \mathcal{D}\big(S(\lambda)\big)\big\}$. Then, if $\lambda \in \sigma_{\mathrm{p}}(T)$ with eigenvector $x \in \mathcal{D}\big(T(\lambda)\big)$, then $\lambda \in \sigma_{\mathrm{p}}(S)$ with eigenvector $W_2(\lambda)x \in \mathcal{D}\big(S(\lambda)\big)$. For the inclusions of the continuous and the residual spectrum, we observe that, due to the assumptions on $W_1$ and $W_2$,

$$R\big(W_1(\lambda)S(\lambda)W_2(\lambda)\big) = W_1(\lambda)R\big(S(\lambda)\big),$$
$$\overline{R\big(W_1(\lambda)S(\lambda)W_2(\lambda)\big)} = W_1(\lambda)\overline{R\big(S(\lambda)\big)}.$$

□

The characterization of the spectra of unbounded block operator matrices by means of their Schur complements seems to have been first used in [Nag89] for the diagonally dominant case; in the upper dominant case, it was also used in [ALMS94] and in subsequent papers to obtain information about the essential spectrum (see Section 2.5).

**Theorem 2.3.3** *If the block operator matrix* $\mathcal{A}$ *satisfies*

i) $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\rho(A) \neq \emptyset$, *and, for some (and hence for all)* $\mu \in \rho(A)$, $(A-\mu)^{-1}B$ *is bounded on* $\mathcal{D}(B)$ *and* $S_2(\mu) = D - \mu - C(A-\mu)^{-1}B$ *is closable, then*

$$\sigma\big(\overline{\mathcal{A}}\big) \setminus \sigma(A) = \sigma\big(\overline{S_2}\big), \qquad \sigma_{\mathrm{p}}\big(\overline{\mathcal{A}}\big) \setminus \sigma(A) = \sigma_{\mathrm{p}}\big(\overline{S_2}\big),$$
$$\sigma_{\mathrm{c}}\big(\overline{\mathcal{A}}\big) \setminus \sigma(A) = \sigma_{\mathrm{c}}\big(\overline{S_2}\big), \qquad \sigma_{\mathrm{r}}\big(\overline{\mathcal{A}}\big) \setminus \sigma(A) = \sigma_{\mathrm{r}}\big(\overline{S_2}\big),$$

*and, for* $\lambda \in \rho\big(\overline{S_2}\big) = \rho\big(\overline{\mathcal{A}}\big) \cap \rho(A) \subset \rho\big(\overline{\mathcal{A}}\big)$,

$$\begin{aligned}
&\big(\overline{\mathcal{A}} - \lambda\big)^{-1} \\
&= \begin{pmatrix} I & -\overline{(A-\lambda)^{-1}B} \\ 0 & I \end{pmatrix} \begin{pmatrix} (A-\lambda)^{-1} & 0 \\ 0 & \overline{S_2(\lambda)}^{\,-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -C(A-\lambda)^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} (A-\lambda)^{-1} + \overline{(A-\lambda)^{-1}B}\,\overline{S_2(\lambda)}^{\,-1} C(A-\lambda)^{-1} & -\overline{(A-\lambda)^{-1}B}\,\overline{S_2(\lambda)}^{\,-1} \\ -\overline{S_2(\lambda)}^{\,-1} C(A-\lambda)^{-1} & \overline{S_2(\lambda)}^{\,-1} \end{pmatrix},
\end{aligned}$$

ii) $\mathcal{D}(D) \subset \mathcal{D}(B)$, $\rho(D) \neq \emptyset$, *and, for some (and hence for all)* $\mu \in \rho(D)$, $(D-\mu)^{-1}C$ *is bounded on* $\mathcal{D}(C)$ *and* $S_1(\mu) = A - \mu - B(D-\mu)^{-1}C$ *is closable, then*

$$\sigma(\overline{\mathcal{A}}) \setminus \sigma(D) = \sigma(\overline{S_1}), \quad \sigma_{\mathrm{p}}(\overline{\mathcal{A}}) \setminus \sigma(D) = \sigma_{\mathrm{p}}(\overline{S_1}),$$
$$\sigma_{\mathrm{c}}(\overline{\mathcal{A}}) \setminus \sigma(D) = \sigma_{\mathrm{c}}(\overline{S_1}), \quad \sigma_{\mathrm{r}}(\overline{\mathcal{A}}) \setminus \sigma(D) = \sigma_{\mathrm{r}}(\overline{S_1}),$$

*and, for* $\lambda \in \rho(\overline{S_1}) = \rho(\overline{\mathcal{A}}) \cap \rho(D) \subset \rho(\overline{\mathcal{A}})$,

$$\begin{aligned}
&(\overline{\mathcal{A}} - \lambda)^{-1} \\
&= \begin{pmatrix} I & 0 \\ -\overline{(D-\lambda)^{-1}C} & I \end{pmatrix} \begin{pmatrix} \overline{S_1(\lambda)}^{-1} & 0 \\ 0 & (D-\lambda)^{-1} \end{pmatrix} \begin{pmatrix} I & -B(D-\lambda)^{-1} \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \overline{S_1(\lambda)}^{-1} & -\overline{S_1(\lambda)}^{-1} B(D-\lambda)^{-1} \\ -\overline{(D-\lambda)^{-1}C}\,\overline{S_1(\lambda)}^{-1} & (D-\lambda)^{-1} + \overline{(D-\lambda)^{-1}C}\,\overline{S_1(\lambda)}^{-1} B(D-\lambda)^{-1} \end{pmatrix}.
\end{aligned}$$

***Proof.*** By the assumptions, the block operator matrix $\mathcal{A}$ satisfies the conditions required for Theorems 2.2.14 and 2.2.18, respectively. All claims in i) and ii) follow from the Frobenius-Schur factorizations (2.2.10) and (2.2.12), respectively, therein together with Lemma 2.3.2. □

For diagonally dominant block operator matrices, the resolvent sets and hence the spectra of the Schur complements can be characterized in terms of the entries of the block operator matrix as follows.

**Proposition 2.3.4** *Suppose that $\mathcal{A}$ is diagonally dominant and closed. If $\rho(A) \cap \rho(D) \neq \emptyset$, then*

$$\rho(S_1) \cap \rho(A) = \{\mu \in \rho(A) \cap \rho(D) : 1 \in \rho\big(B(D-\mu)^{-1}C(A-\mu)^{-1}\big)\},$$
$$\rho(S_2) \cap \rho(D) = \{\mu \in \rho(A) \cap \rho(D) : 1 \in \rho\big(C(A-\mu)^{-1}B(D-\mu)^{-1}\big)\};$$

*moreover, $\rho(S_1) \cap \rho(A)$, $\rho(S_2) \cap \rho(D)$ are open.*

***Proof.*** We prove the claims for $S_1$; the proof for $S_2$ is analogous. Since $S_1$ is defined on $\rho(D)$, we have $\rho(S_1) \subset \rho(D)$. For $\mu \in \rho(A) \cap \rho(D)$, we can write

$$S_1(\mu) = M(\mu)(A-\mu), \quad M(\mu) := I - B(D-\mu)^{-1}C(A-\mu)^{-1}.$$

The operators $A$, $D$ are closed because $\rho(A), \rho(D) \neq \emptyset$. Since $\mathcal{A}$ is diagonally dominant, the operators $B(D-\mu)^{-1}$, $C(A-\mu)^{-1}$ are everywhere defined and bounded (see Remark 2.2.2 i)), thus closed, and hence so is $M(\mu)$. Clearly, if $M(\mu)$ is boundedly invertible, then so is $S_1(\mu)$. Vice versa, if $\mu \in \rho(S_1)$, then $M(\mu)$ is bijective. Since $M(\mu)$ is closed, it is thus boundedly invertible by the closed graph theorem.

To prove that $\rho(S_1) \cap \rho(A)$ is open, let $\mu_0 \in \rho(S_1) \cap \rho(A) \subset \rho(A) \cap \rho(D)$. Then $M(\mu_0)$ is boundedly invertible. Choose $\varepsilon > 0$ such that $\{\mu \in \mathbb{C} : |\mu-\mu_0| \leq \varepsilon\} \subset \rho(A) \cap \rho(D)$. Then, for $\mu \in \mathbb{C}$, $|\mu-\mu_0| \leq \varepsilon$, we have

$$\begin{aligned}\big\|M(\mu_0) - M(\mu)\big\| &= \big\|\big(B(D-\mu_0)^{-1} - B(D-\mu)^{-1}\big)C(A-\mu_0)^{-1} \\ &\qquad + B(D-\mu)^{-1}\big(C(A-\mu_0)^{-1} - C(A-\mu)^{-1}\big)\big\| \\ &\leq |\mu-\mu_0|\,\big(\|B(D-\mu_0)^{-1}\|\,\|(D-\mu)^{-1}\|\,\|C(A-\mu_0)^{-1}\| \\ &\qquad + \|B(D-\mu)^{-1}\|\,\|C(A-\mu_0)^{-1}\|\,\|(A-\mu)^{-1}\|\big) \\ &\leq |\mu-\mu_0|\,C_\varepsilon\end{aligned}$$

with some constant $C_\varepsilon > 0$. Here we have used the fact that the mappings $\lambda \mapsto (A-\lambda)^{-1}$, $\lambda \mapsto (D-\lambda)^{-1}$, and hence $\lambda \mapsto B(D-\lambda)^{-1}$ are holomorphic on $\rho(A)\cap\rho(D)$ with values in $L(\mathcal{H}_1)$, $L(\mathcal{H}_2)$, and $L(\mathcal{H}_2,\mathcal{H}_1)$, respectively; in the last case, this follows from the relation

$$B(D-\lambda)^{-1} = B(D-\mu_0)^{-1} + (\lambda-\mu_0)B(D-\mu_0)^{-1}(D-\lambda)^{-1}, \quad \lambda\in\rho(D).$$

For $\mu\in\mathbb{C}$, $|\mu-\mu_0| < \min\{\varepsilon, 1/C_\varepsilon\}$, a Neumann series argument implies that $M(\mu)$ is boundedly invertible and hence $\mu\in\rho(S_1)\cap\rho(A)$. □

The following corollary is immediate from Theorem 2.3.3 i) and Proposition 2.3.4; it was first proved in [Nag89, Lemma 2.1], [Nag97, Theorem 2.4].

**Corollary 2.3.5** *Suppose that $\mathcal{A}$ is diagonally dominant and closed. Then, for $\lambda\in\rho(A)\cap\rho(D)$, the following are equivalent:*

i) $\lambda\in\rho(\mathcal{A})$,
ii) $1\in\rho\big(B(D-\lambda)^{-1}C(A-\lambda)^{-1}\big)$,
iii) $1\in\rho\big(C(A-\lambda)^{-1}B(D-\lambda)^{-1}\big)$.

Next we use Theorems 2.2.14 and 2.3.3 i) to establish a criterion for symmetric block operator matrices in a Hilbert space to be essentially self-adjoint, *i.e.* $\overline{\mathcal{A}} = \overline{\mathcal{A}}^*$; the formulation of the analogous result obtained from Theorems 2.2.18 and 2.3.3 ii) is left to the reader.

**Proposition 2.3.6** *Suppose that $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, $A$, $D$ are self-adjoint, and $C = B^*$. If $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$ and $\mathcal{D}(B)\cap\mathcal{D}(D)$ is a core of $D$, then $\mathcal{A}$ is essentially self-adjoint and*

$$\mathcal{D}\big(\overline{\mathcal{A}}\big) = \left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathcal{H}\colon\, x + \overline{(A-\mu)^{-1}B}\,y\in\mathcal{D}(A),\, y\in\mathcal{D}(D)\right\},$$

$$\overline{\mathcal{A}}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}(A-\mu)\big(x+\overline{(A-\mu)^{-1}B}\,y\big)+\mu\,x\\ B^*x+Dy\end{pmatrix},$$

*where $\mu\in\rho(A)$ is arbitrary; in particular, $\mathcal{D}\big(\overline{\mathcal{A}}\big)\subset\mathcal{D}(|A|^{1/2})\oplus\mathcal{D}(D)$.*

***Proof.*** First we show that the assumptions of Theorem 2.3.3 i) are satisfied. Since $\mathcal{D}(A) = \mathcal{D}(|A|)\subset\mathcal{D}(|A|^{1/2})$, we have $\mathcal{D}(A)\subset\mathcal{D}(B^*) = \mathcal{D}(C)$.

Further, $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$ implies that, for every $\mu \in \rho(A)$, the operator $B^*(A-\mu)^{-1/2}$ is bounded and everywhere defined and the operator $(A-\mu)^{-1/2}B \subset \big(B^*(A-\mu)^{-1/2}\big)^*$ is bounded and densely defined with closure $\overline{(A-\mu)^{-1/2}B} = \big(B^*(A-\mu)^{-1/2}\big)^*$. Thus $(A-\mu)^{-1}B = (A-\mu)^{-1/2}(A-\mu)^{-1/2}B$ is bounded on $\mathcal{D}(B)$, and $B^*(A-\mu)^{-1}B = B^*(A-\mu)^{-1/2}(A-\mu)^{-1/2}B$ is closable, being bounded on the dense set $\mathcal{D}(B)$.

Obviously, $\mathcal{A}$ is symmetric. In order to show that $\mathcal{A}$ is essentially self-adjoint, it suffices to prove that there exist $\mu_1, \mu_2 \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Im}\mu_1 > 0$, $\operatorname{Im}\mu_2 < 0$, such that $\mu_1, \mu_2 \in \rho\big(\overline{S_2}\big)$ and hence $\mu_1, \mu_2 \in \rho\big(\overline{\mathcal{A}}\big)$ by Theorem 2.3.3 i). Since $\mathcal{D}(B) \cap \mathcal{D}(D)$ is a core of $D$, we have, for $\mu \in \mathbb{C} \setminus \mathbb{R}$,

$$\overline{S_2(\mu)} = D - \mu - \overline{B^*(A-\mu)^{-1}B} = (D-\mu)\big(I - (D-\mu)^{-1}\overline{B^*(A-\mu)^{-1}B}\big).$$

Now choose $\mu_1, \mu_2 \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Im}\mu_1 > 0$, $\operatorname{Im}\mu_2 < 0$, such that

$$|\operatorname{Im}\mu_i| > \big\|\overline{B^*(A-\mu)^{-1}B}\big\|, \quad i = 1, 2.$$

Then a Neumann series argument shows that $\mu_1, \mu_2 \in \rho\big(\overline{S_2}\big)$.

The formula for $\mathcal{D}\big(\overline{\mathcal{A}}\big)$ follows from Theorem 2.2.14 if we observe that $\mathcal{D}\big(\overline{S_2(\mu)}\big) = \mathcal{D}(D)$ for every $\mu \in \rho(A)$. Since the operator $(A-\mu)^{-1}B$ in (2.2.10) is bounded and densely defined with closure $\overline{(A-\mu)^{-1}B} = \overline{(A-\mu)^{-1/2}(A-\mu)^{-1/2}B} = (A-\mu)^{-1/2}\overline{(A-\mu)^{-1/2}B}$, we conclude that $\mathcal{D}\big(\overline{\mathcal{A}}\big) \subset \mathcal{D}(|A|^{1/2}) \oplus \mathcal{D}(D) \subset \mathcal{D}(B^*) \oplus \mathcal{D}(D)$. Now the representation of $\overline{\mathcal{A}}$ is a direct consequence of Theorem 2.2.14. □

The quadratic complements may also be used to describe the spectrum and the resolvent of block operator matrices.

**Theorem 2.3.7** *If the block operator matrix $\mathcal{A}$ satisfies*

i) *$\mathcal{D}(C) \subset \mathcal{D}(A)$, $C$ is boundedly invertible, $C^{-1}D$ is bounded on $\mathcal{D}(D)$, and, for some (and hence for all) $\mu \in \mathbb{C}$, $T_2(\mu) = B - (A-\mu)C^{-1}(D-\mu)$ is closable, then*

$$\sigma\big(\overline{\mathcal{A}}\big) = \sigma\big(\overline{T_2}\big), \quad \sigma_{\mathrm{p}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{p}}\big(\overline{T_2}\big),$$
$$\sigma_{\mathrm{c}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{c}}\big(\overline{T_2}\big), \quad \sigma_{\mathrm{r}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{r}}\big(\overline{T_2}\big),$$

*and, for $\lambda \in \rho\big(\overline{\mathcal{A}}\big) = \rho\big(\overline{T_2}\big)$,*

$$\begin{aligned}&\big(\overline{\mathcal{A}} - \lambda\big)^{-1}\\ &= \begin{pmatrix} I & -\overline{C^{-1}(D-\lambda)} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & C^{-1} \\ \overline{T_2(\lambda)}^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & -(A-\lambda)C^{-1} \\ 0 & I \end{pmatrix}\\ &= \begin{pmatrix} -\overline{C^{-1}(D-\lambda)}\,\overline{T_2(\lambda)}^{-1} & C^{-1} + \overline{C^{-1}(D-\lambda)}\,\overline{T_2(\lambda)}^{-1}(A-\lambda)C^{-1} \\ \overline{T_2(\lambda)}^{-1} & -\overline{T_2(\lambda)}^{-1}(A-\lambda)C^{-1} \end{pmatrix},\end{aligned}$$

ii) $\mathcal{D}(B) \subset \mathcal{D}(D)$, *$B$ is boundedly invertible,* $B^{-1}A$ *is bounded on* $\mathcal{D}(A)$, *and, for some (and hence for all)* $\mu \in \mathbb{C}$, $T_1(\mu) = C-(D-\mu)B^{-1}(A-\mu)$ *is closable, then*

$$\sigma\big(\overline{\mathcal{A}}\big) = \sigma\big(\overline{T_1}\big), \quad \sigma_{\mathrm{p}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{p}}\big(\overline{T_1}\big),$$
$$\sigma_{\mathrm{c}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{c}}\big(\overline{T_1}\big), \quad \sigma_{\mathrm{r}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{r}}\big(\overline{T_1}\big),$$

*and, for* $\lambda \in \rho\big(\overline{\mathcal{A}}\big) = \rho\big(\overline{T_1}\big)$,

$$\begin{aligned}&\big(\overline{\mathcal{A}} - \lambda\big)^{-1}\\ &= \begin{pmatrix} I & 0 \\ -\overline{B^{-1}(A-\lambda)} & I \end{pmatrix} \begin{pmatrix} 0 & \overline{T_1(\lambda)}^{-1} \\ B^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -(D-\lambda)B^{-1} & I \end{pmatrix}\\ &= \begin{pmatrix} -\overline{T_1(\lambda)}^{-1}(D-\lambda)B^{-1} & \overline{T_1(\lambda)}^{-1} \\ B^{-1} + \overline{B^{-1}(A-\lambda)}\,\overline{T_1(\lambda)}^{-1}(D-\lambda)B^{-1} & -\overline{B^{-1}(A-\lambda)}\,\overline{T_1(\lambda)}^{-1} \end{pmatrix}.\end{aligned}$$

***Proof.*** By the assumptions, $\mathcal{A}$ satisfies the conditions of Theorems 2.2.23 and 2.2.25, respectively. Thus all claims follow from the factorizations (2.2.15) and (2.2.17), respectively, therein together with Lemma 2.3.2. □

The following characterization of the spectrum of an off-diagonally dominant block operator matrix is analogous to Corollary 2.3.5.

**Corollary 2.3.8** *Suppose that $\mathcal{A}$ is off-diagonally dominant and closed, and that $B$, $C$ are boundedly invertible. Then the following are equivalent:*

i) $\lambda \in \rho(\mathcal{A})$,
ii) $1 \in \rho\big((A-\lambda)C^{-1}(D-\lambda)B^{-1}\big)$,
iii) $1 \in \rho\big((D-\lambda)B^{-1}(A-\lambda)C^{-1}\big)$.

## 2.4 The essential spectrum

The essential spectrum is not as widely known as other parts of the spectrum, in particular, eigenvalues. Nevertheless, information about it is indispensable in applications for various reasons: the stability of a physical system is guaranteed only if the whole spectrum lies in a half-plane, and, near the essential spectrum, numerical calculations of eigenvalues become difficult. Since the essential spectrum itself is hardly accessible by numerical methods, it has to be tackled analytically.

If the essential spectrum of a closed linear operator is empty and its resolvent set is non-empty, then the spectrum consists only of isolated eigenvalues of finite algebraic multiplicity which accumulate at most at $\infty$. This is a typical situation for regular differential operators. For regular *matrix* differential operators, however, the essential spectrum need not be empty; a simple example is given in Example 2.4.3 below.

In this section we establish some abstract results that allow us to determine the essential spectrum of block operator matrices. The first method relies on perturbation arguments; the second method uses the previous factorization results which relate the essential spectrum of a block operator matrix to that of its Schur complements or quadratic complements.

The following perturbation theorem was first proved in [MT94] for diagonally dominant block operator matrices.

**Theorem 2.4.1** *Let $A_i : \mathcal{H}_1 \to \mathcal{H}_1$, $B_i : \mathcal{H}_2 \to \mathcal{H}_1$, $C_i : \mathcal{H}_1 \to \mathcal{H}_2$, and $D_i : \mathcal{H}_2 \to \mathcal{H}_2$, $i = 0, 1$, be unbounded closable operators such that*

$$\mathcal{D}(A_0) \subset \mathcal{D}(A_1), \quad \mathcal{D}(B_0) \subset \mathcal{D}(B_1), \quad \mathcal{D}(C_0) \subset \mathcal{D}(C_1), \quad \mathcal{D}(D_0) \subset \mathcal{D}(D_1),$$

*and define the block operator matrices*

$$\mathcal{A}_0 := \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \quad \mathcal{A}_1 := \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

*Suppose that either*

i) *$C_0$ is $A_0$-bounded with $A_0$-bound $\delta_{C_0}$ and $B_0$ is $D_0$-bounded with $D_0$-bound $\delta_{B_0}$ such that $\delta_{B_0}\delta_{C_0} < 1$,*

ii) *$A_1$, $C_1$ are $A_0$-compact and $B_1$, $D_1$ are $D_0$-compact,*

*or that*

iii) *$A_0$ is $C_0$-bounded with $C_0$-bound $\delta_{A_0}$ and $D_0$ is $B_0$-bounded with $B_0$-bound $\delta_{D_0}$ such that $\delta_{A_0}\delta_{D_0} < 1$,*

iv) *$A_1$, $C_1$ are $C_0$-compact and $B_1$, $D_1$ are $B_0$-compact.*

*Then $\mathcal{A}_1$ is $\mathcal{A}_0$-compact; if $\mathcal{A}_0$ is closed, then $\mathcal{A} := \mathcal{A}_0 + \mathcal{A}_1$ is closed and*

$$\sigma_{\mathrm{ess}}(\mathcal{A}_0 + \mathcal{A}_1) = \sigma_{\mathrm{ess}}(\mathcal{A}_0).$$

***Proof.*** We consider the second case where iii) and iv) hold and $\mathcal{A}$ is off-diagonally dominant; the proof for the first case where $\mathcal{A}$ is diagonally dominant is completely analogous and may be found in [MT94, Theorem 2.2].

Let $(\mathbf{f}_n)_1^\infty \subset \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(C_0) \oplus \mathcal{D}(B_0)$, $\mathbf{f}_n = (f_n \ g_n)^{\mathrm{t}}$, be such that $(\mathbf{f}_n)_1^\infty$, $(\mathcal{A}_0\mathbf{f}_n)_1^\infty$ are bounded. Then we have, for $n \in \mathbb{N}$,

$$\begin{aligned} A_0 f_n + B_0 g_n &= h_n, \\ C_0 f_n + D_0 g_n &= k_n \end{aligned}$$

with bounded sequences $(h_n)_1^\infty \subset \mathcal{H}_1$, $(k_n)_1^\infty \subset \mathcal{H}_2$. Choose $\varepsilon > 0$ so that $(\delta_{A_0}+\varepsilon)(\delta_{D_0}+\varepsilon) < 1$. By assumption iii), there exist constants $a_{A_0}$, $a_{D_0}$, $b_{A_0}$, $b_{D_0} \geq 0$ with $\delta_{A_0} \leq b_{A_0} < \delta_{A_0} + \varepsilon$, $\delta_{D_0} \leq b_{D_0} < \delta_{D_0} + \varepsilon$ and

$$\|A_0 f\| \leq a_{A_0}\|f\| + b_{A_0}\|C_0 f\|, \quad f \in \mathcal{D}(C_0),$$
$$\|D_0 g\| \leq a_{D_0}\|g\| + b_{D_0}\|B_0 g\|, \quad g \in \mathcal{D}(B_0).$$

We conclude that, for $n \in \mathbb{N}$,

$$\begin{aligned}\|B_0 g_n\| &\leq \|h_n\| + \|A_0 f_n\| \\ &\leq \|h_n\| + a_{A_0}\|f_n\| + b_{A_0}\|C_0 f_n\| \\ &\leq \|h_n\| + a_{A_0}\|f_n\| + b_{A_0}\big(\|k_n\| + \|D_0 g_n\|\big) \\ &\leq \|h_n\| + a_{A_0}\|f_n\| + b_{A_0}\|k_n\| + a_{D_0} b_{A_0}\|g_n\| + b_{A_0} b_{D_0}\|B_0 g_n\|.\end{aligned}$$

Since $b_{A_0} b_{D_0} < 1$, it follows that

$$\|B_0 g_n\| \leq \frac{\|h_n\| + a_{A_0}\|f_n\| + b_{A_0}\|k_n\| + a_{D_0} b_{A_0}\|g_n\|}{1 - b_{A_0} b_{D_0}}$$

and hence $(B_0 g_n)_1^\infty \subset \mathcal{H}_1$ is a bounded sequence. Analogously, one can prove that $(C_0 f_n)_1^\infty \subset \mathcal{H}_2$ is bounded. Since $A_1$, $C_1$ are $C_0$-compact and $B_1$, $D_1$ are $B_0$-compact, there exist subsequences $(f_{n_k})_{k=1}^\infty$ and $(g_{n_k})_{k=1}^\infty$ such that $(A_1 f_{n_k})_{k=1}^\infty$, $(C_1 f_{n_k})_{k=1}^\infty$, $(B_1 g_{n_k})_{k=1}^\infty$, and $(D_1 g_{n_k})_{k=1}^\infty$ are convergent. Consequently, for the subsequence $(\mathbf{f}_{n_k})_{k=1}^\infty$ with $\mathbf{f}_{n_k} := (f_{n_k}\ g_{n_k})^{\mathrm{t}}$, the sequence $(\mathcal{A}_1 \mathbf{f}_{n_k})_{k=1}^\infty \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ with elements

$$\mathcal{A}_1 \mathbf{f}_{n_k} = \begin{pmatrix} A_1 f_{n_k} + B_1 g_{n_k} \\ C_1 f_{n_k} + D_1 g_{n_k} \end{pmatrix}, \quad k \in \mathbb{N},$$

converges as well. This proves that $\mathcal{A}_1$ is $\mathcal{A}_0$-compact. The last statement follows from Theorem 2.1.13. □

**Remark 2.4.2** The assumptions i) and iii) of Theorem 2.4.1 are satisfied if the following conditions i′) and iii′), respectively, hold:

- i′) $A_0$, $D_0$ are closed and either $\mathcal{D}(C_0) \subset \mathcal{D}(A_0)$ and $B_0$ is bounded or $\mathcal{D}(B_0) \subset \mathcal{D}(D_0)$ and $C_0$ is bounded;
- iii′) $B_0$, $C_0$ are closed and either $\mathcal{D}(A_0) \subset \mathcal{D}(C_0)$ and $D_0$ is bounded or $\mathcal{D}(D_0) \subset \mathcal{D}(B_0)$ and $A_0$ is bounded.

Then, in each case, one of the relative bounds is 0 and so $\mathcal{A}_0$ is automatically closed by Corollary 2.2.9. In the particular case $B_0 = 0$ or $C_0 = 0$ of i′), the block operator matrix $\mathcal{A}_0$ is upper or lower block triangular and hence

$$\sigma_{\mathrm{ess}}(\mathcal{A}_0 + \mathcal{A}_1) = \sigma_{\mathrm{ess}}(\mathcal{A}_0) = \sigma_{\mathrm{ess}}(A_0) \cup \sigma_{\mathrm{ess}}(D_0).$$

As a first simple illustration of the above perturbation results, we consider block operator matrices that arise as linearizations of $\lambda$-rational Sturm-Liouville problems, which exhibit a so-called "floating singularity" (see [Bog85], [LMM90], [AL95]).

**Example 2.4.3** Let $q \in L_1(0,1)$ and $w, u \in L_\infty[0,1]$. In the Hilbert space $L_2(0,1)$, we consider the spectral problem

$$\left(-\mathrm{D}^2 + q - \lambda - \frac{w}{u-\lambda}\right) y = 0, \quad y(0) = y(1) = 0, \tag{2.4.1}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}x$ is the derivative with respect to the variable $x \in [0,1]$. Depending on information about the sign of the function $w$, there are different ways of transforming (2.4.1) into a spectral problem for a block operator matrix. We can write

$$w = w_1 w_2, \quad \begin{cases} w_1 = \sqrt{w}, & w_2 = \sqrt{w} & \text{if } \ w \geq 0, \\ w_1 = \sqrt{|w|}, & w_2 = -\sqrt{|w|} & \text{if } \ w \leq 0, \\ w_1 = 1, & w_2 = w & \text{in any case.} \end{cases}$$

According to this factorization, we set

$$y_1 := y, \quad y_2 := \frac{w_2}{u-\lambda} y.$$

Then (2.4.1) is equivalent to the spectral problem

$$\begin{pmatrix} -\mathrm{D}^2 + q & w_1 \\ w_2 & u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1(0) = y_1(1) = 0, \tag{2.4.2}$$

in $L_2(0,1) \oplus L_2(0,1)$. Here the block operator matrix $\mathcal{A}$ with entries

$$\begin{aligned} &A := -\mathrm{D}^2 + q, && \mathcal{D}(A) := \big\{ y_1 \in W_2^2(0,1) : y_1(0) = y_1(1) = 0 \big\}, \\ &B := w_1 \,\cdot\,, \ C := w_2 \,\cdot\,, && \mathcal{D}(B) := \mathcal{D}(C) := L_2(0,1), \\ &D := u \,\cdot\,, && \mathcal{D}(D) := L_2(0,1), \end{aligned}$$

can be decomposed as

$$\mathcal{A} = \begin{pmatrix} -\mathrm{D}^2 + q & w_1 \\ 0 & u \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ w_2 & 0 \end{pmatrix} =: \mathcal{A}_0 + \mathcal{A}_1.$$

Without loss of generality, we may assume that $0 \notin \mathcal{R}(u)$ (the essential range of the function $u$); otherwise we shift the spectral parameter correspondingly. Then the above decomposition of $\mathcal{A}$ satisfies all assumptions of Theorem 2.4.1 (see Remark 2.4.2). In fact, the operator $A_0 = -\mathrm{D}^2 + q$ with Dirichlet boundary conditions on $[0,1]$ has compact resolvent since $q \in L_1(0,1)$ implies that $q$ is relatively form-bounded with form-bound 0 with respect to $-\mathrm{D}^2$ (see [BS87, Section 10.5.4], [RS78, Theorem XIII.68]);

thus the bounded multiplication operator $w_2\,\cdot$ is relatively compact with respect to $A_0$; moreover, $w_1\,\cdot$ is relatively bounded with respect to the boundedly invertible multiplication operator $u\,\cdot$. Now Remark 2.4.2 yields that the essential spectrum of $\mathcal{A}$ (*i.e.* that of (2.4.2)) is given by

$$\sigma_{\mathrm{ess}}(\mathcal{A}) = \sigma_{\mathrm{ess}}(\mathcal{A}_0) = \sigma_{\mathrm{ess}}(u\,\cdot) = \mathcal{R}(u);$$

in particular, if $u$ is continuous on $[0,1]$, then $\sigma_{\mathrm{ess}}(\mathcal{A}) = u\big([0,1]\big)$.

Note that, in fact, the original $\lambda$-rational spectral problem (2.4.1) is the spectral problem for the first Schur complement $S_1$ of the block operator matrix $\mathcal{A}$. In [LMM90] and all subsequent papers, the formula for the essential spectrum was obtained using the second Schur complement $S_2$ (see also Remark 2.4.11), not by the simple perturbation argument above.

**Remark 2.4.4** The above result and its proof carry over if one replaces $-\mathrm{D}^2$ by an arbitrary elliptic differential operator of order $2m$ on a bounded region $\Omega \subset \mathbb{R}^n$ of class $C^m$ with suitable boundary conditions so that the resolvent is compact and if $q, w, u$ are bounded functions on $\Omega$ (see [FM91]).

An example for the off-diagonally dominant case is furnished by Dirac operators in $\mathbb{R}^3$, which describe the behaviour of a quantum mechanical particle of spin 1/2 (see [Tha92] and Section 3.3.1).

**Example 2.4.5** Denote by $m$ and $e$ the mass and the charge, respectively, of a relativistic spin 1/2 particle, by $c$ the velocity of light, by $\hbar$ the Planck constant, and by $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the vector of the Pauli spin matrices

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a scalar potential and $\vec{A} = (a_1, a_2, a_3) : \mathbb{R}^3 \to \mathbb{R}^3$ a vector potential generating an electric field $\vec{E} = \nabla\phi$ and a magnetic field $\vec{B} = \operatorname{rot}\vec{A}$, respectively. Then the Dirac operator in $\mathbb{R}^3$ with electromagnetic potential is the block operator matrix (see [Tha92, (4.14)])

$$\mathbf{H}_{\boldsymbol{\Phi}} = \begin{pmatrix} mc^2 + e\phi & c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big) \\ c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big) & -mc^2 + e\phi \end{pmatrix} \tag{2.4.3}$$

in $L_2(\mathbb{R}^3)^4 = L_2(\mathbb{R}^3)^2 \oplus L_2(\mathbb{R}^3)^2$. For $\phi = 0$, $\vec{A} = 0$, the free Dirac operator

$$\mathbf{H}_{\mathrm{free}} = \begin{pmatrix} mc^2 & c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla\big) \\ c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla\big) & -mc^2 \end{pmatrix}, \tag{2.4.4}$$

is self-adjoint on $\mathcal{D}(\mathbf{H}_{\mathrm{free}}) = H^1(\mathbb{R}^3)^4 = H^1(\mathbb{R}^3)^2 \oplus H^1(\mathbb{R}^3)^2$, where $H^1(\mathbb{R}^3)$ is the first order Sobolev space. According to Theorem 2.4.1, we decompose

$$\mathbf{H_\Phi} = \begin{pmatrix} mc^2 & c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big) \\ c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big) & -mc^2 \end{pmatrix} + \begin{pmatrix} e\phi & 0 \\ 0 & e\phi \end{pmatrix} =: \mathbf{H_0} + \mathbf{V}.$$

The Dirac operator $\mathbf{H_\Phi}$ is self-adjoint on $\mathcal{D}(\mathbf{H_0}) = \mathcal{H}_1(\vec{A}) \oplus \mathcal{H}_1(\vec{A})$ with

$$\mathcal{H}_1(\vec{A}) := \Big\{ y \in L_2(\mathbb{R}^3)^2 : \Big(\mathrm{i}\hbar\,\partial_\nu + \frac{e}{c}a_\nu\Big)y \in L_2(\mathbb{R}^3)^2,\ \nu = 1,2,3 \Big\}$$

if the potential $V\cdot = e\phi\cdot$ is $\big(c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big)\big)$-bounded with relative bound $< 1$. Then assumption iii) of Theorem 2.4.1 holds with $\delta_{A_0} = \delta_{D_0} = 0$ as $A_0 = mc^2$, $D_0 = -mc^2$ are both bounded (see Remark 2.4.2). If the potential $V\cdot = e\phi\cdot$ is $\big(c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big)\big)$-compact, then assumption iv) of Theorem 2.4.1 holds and we obtain, using the properties of the Pauli matrices $\vec{\sigma}$ (see [Tha92, (5.79)])

$$\sigma_{\mathrm{ess}}(\mathbf{H_\Phi}) = \sigma_{\mathrm{ess}}(\mathbf{H_0}) = \Big\{ \lambda \in \mathbb{R} : \lambda^2 - (mc^2)^2 \in \sigma_{\mathrm{ess}}\Big( c^2\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big)^2 - ec\,\vec{\sigma}\cdot\vec{B} \Big) \Big\}.$$

The relative compactness assumption on $V$ and hence the relative boundedness assumption is satisfied if, for example,

$$\vec{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)^3, \quad \|\vec{A}\| \in L_\infty(\mathbb{R}^3), \quad \|\vec{B}\| \in L_{3/2}(\mathbb{R}^3), \quad V \in L_3(\mathbb{R}^3)$$

(see [EL99, Theorems 2.2, 2.4, Corollary 2.3]). Moreover, if $V \in L_3(\mathbb{R}^3)$ and either $\vec{A} \in C^\infty(\mathbb{R}^3)^3$ and $\big|\vec{B}(x)\big| \to 0$, $|x| \to \infty$, or $\vec{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^3)^3$ and $\|\vec{B}\| \in L_2(\mathbb{R}^3)$, then

$$\sigma_{\mathrm{ess}}(\mathbf{H_\Phi}) = \sigma_{\mathrm{ess}}(\mathbf{H_0}) = \sigma_{\mathrm{ess}}(\mathbf{H}_{\mathrm{free}}) = (-\infty, -mc^2] \cup [mc^2, \infty) \tag{2.4.5}$$

(see [EL99, p. 190] and compare [Tha92, Section 4.3.4]); in particular, this holds for $\vec{A} = 0$ and $V \in L_3(\mathbb{R}^3)$.

In the remaining part of this section, we use the factorizations (2.2.10), (2.2.12) and (2.2.15), (2.2.17) in order to express the essential spectrum of a block operator matrix in terms of the essential spectra of the Schur complements $S_1$, $S_2$ and of the quadratic complements $T_1$, $T_2$.

In the following, if $E, F$ are Banach spaces and $S$ is an operator function defined on some subset $\Omega \subset \mathbb{C}$ such that $S(\lambda)$ are linear operators from $E$ to $F$, we introduce the essential spectrum of $S$ as

$$\sigma_{\mathrm{ess}}(S) := \big\{ \lambda \in \Omega : 0 \in \sigma_{\mathrm{ess}}\big(S(\lambda)\big) \big\}. \tag{2.4.6}$$

Since Fredholm operators are closed, $\lambda \in \sigma_{\mathrm{ess}}(S)$ if $S(\lambda)$ is not closed. If $S(\lambda)$ is a closable for all $\lambda \in \Omega$, we define the operator function $\overline{S}$ by

$$\overline{S}(\lambda) := \overline{S(\lambda)}, \quad \lambda \in \Omega.$$

**Theorem 2.4.6** *Suppose that $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $\rho(A) \neq \emptyset$. If, for some* (*and hence for all*) *$\mu \in \rho(A)$,*

i) *$(A-\mu)^{-1}B$ is bounded on $\mathcal{D}(B)$,*
ii) *$S_2(\mu) = D - \mu - C(A-\mu)^{-1}B$ is closable,*

*then*

$$\sigma_{\text{ess}}\big(\overline{\mathcal{A}}\big) \setminus \sigma(A) = \sigma_{\text{ess}}\big(\overline{S_2}\big).$$

***Proof.*** In the factorization (2.2.10), the first and last factor are bounded and boundedly invertible and $A-\lambda$ is bijective for $\lambda \notin \sigma(A)$. Thus $\overline{\mathcal{A}} - \lambda$ is Fredholm if and only if so is $\overline{S_2(\lambda)}$ by [GGK90, Theorem XVII.3.1]. □

If $\mathcal{D}(D) \subset \mathcal{D}(B)$, then the essential spectrum of $\mathcal{A}$ may be described in terms of the first Schur complement; the corresponding analogue of Theorem 2.4.6 is formulated below. If $\mathcal{A}$ is diagonally dominant, we may choose the Schur complement whose essential spectrum is easier to calculate.

**Theorem 2.4.7** *Suppose that $\mathcal{D}(D) \subset \mathcal{D}(B)$ and $\rho(D) \neq \emptyset$. If, for some* (*and hence for all*) *$\mu \in \rho(D)$,*

i) *$(D-\mu)^{-1}C$ is bounded on $\mathcal{D}(C)$,*
ii) *$S_1(\mu) = A - \mu - B(D-\mu)^{-1}C$ is closable,*

*then*

$$\sigma_{\text{ess}}\big(\overline{\mathcal{A}}\big) \setminus \sigma(D) = \sigma_{\text{ess}}\big(\overline{S_1}\big).$$

Under certain compactness assumptions, the description of the essential spectrum of a block operator matrix $\mathcal{A}$ can be further improved. The following result generalizes a corresponding result in [Kon98, Theorem 1] for self-adjoint block operator matrices; the proof relies on the fact that the difference of the resolvents of $\overline{\mathcal{A}}$ and of the block diagonal matrix

$$\overline{\mathcal{A}}_{1,\mu_0} := \begin{pmatrix} A & 0 \\ 0 & \overline{D - C(A-\mu_0)^{-1}B} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \overline{S_2(\mu_0)} + \mu_0 \end{pmatrix} \tag{2.4.7}$$

for an arbitrary fixed $\mu_0 \in \rho(A)$ is compact.

**Theorem 2.4.8** *Let $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\rho(A) \cap \rho(D) \neq \emptyset$, and let $\mathcal{D}(B) \cap \mathcal{D}(D)$ be a core of $D$. If, for some* (*and hence for all*) *$\mu \in \rho(A) \cap \rho(D)$,*

i) *$(A-\mu)^{-1}B$ and $C(A-\mu)^{-1}B$ are bounded on $\mathcal{D}(B)$,*
ii) *$(D-\mu)^{-1}C(A-\mu)^{-1}$ and $(A-\mu)^{-1}B(D-\mu)^{-1}$ are compact,*

*then, for every $\mu_0 \in \rho(A)$ with $\rho\big(\overline{\mathcal{A}}\big) \cap \rho(A) \cap \rho\big(D - \overline{C(A-\mu_0)^{-1}B}\big) \neq \emptyset$, the difference of the resolvents $\big(\overline{\mathcal{A}} - \lambda\big)^{-1}$ and $\big(\overline{\mathcal{A}}_{1,\mu_0} - \lambda\big)^{-1}$ is compact for $\lambda \in \rho\big(\overline{\mathcal{A}}\big) \cap \rho(A) \cap \rho\big(D - \overline{C(A-\mu_0)^{-1}B}\big)$; in particular,*

$$\sigma_{\text{ess}}\big(\overline{\mathcal{A}}\big) = \sigma_{\text{ess}}(A) \cup \sigma_{\text{ess}}\big(D - \overline{C(A-\mu_0)^{-1}B}\big).$$

***Proof.*** Since $\rho(A)\cap\rho(D)\neq\emptyset$, the operators $A$ and $D$ are closed. By the second condition in i), $S_2(\mu)$ is closable for every $\mu\in\rho(A)$ and

$$\overline{S_2(\mu)} = D-\mu-\overline{C(A-\mu)^{-1}B} \tag{2.4.8}$$

because $\mathcal{D}(B)\cap\mathcal{D}(D)$ is a core of $D$. Together with the first condition in i), Theorem 2.2.14 shows that the block operator matrix $\mathcal{A}$ is closable. Now let $\mu_0\in\rho(A)$ and $\lambda\in\mathbb{C}$ be so that $\lambda\in\rho\big(\overline{\mathcal{A}}\big)\cap\rho(A)\cap\rho\big(D-\overline{C(A-\mu_0)^{-1}B}\big)\neq\emptyset$. Using the formula for the resolvent of $\overline{\mathcal{A}}$ in Theorem 2.3.3 i), we obtain that

$$\begin{aligned}&\big(\overline{\mathcal{A}}-\lambda\big)^{-1}-\big(\overline{\mathcal{A}}_{1,\mu_0}-\lambda\big)^{-1}\\&=\begin{pmatrix}\overline{(A-\lambda)^{-1}B}\;\overline{S_2(\lambda)}^{\,-1}C(A-\lambda)^{-1} & -\overline{(A-\lambda)^{-1}B}\;\overline{S_2(\lambda)}^{\,-1}\\ -\overline{S_2(\lambda)}^{\,-1}C(A-\lambda)^{-1} & \overline{S_2(\lambda)}^{\,-1}-\big(\overline{S_2(\mu_0)}+\mu_0-\lambda\big)^{-1}\end{pmatrix}.\end{aligned}$$

It remains to be shown that all entries of this block operator matrix are compact. For the left lower corner, we observe that, by (2.4.8),

$$\begin{aligned}&\overline{S_2(\lambda)}^{\,-1}C(A-\lambda)^{-1}\\&=(D-\lambda)^{-1}C(A-\lambda)^{-1}+\big(\overline{S_2(\lambda)}^{\,-1}-(D-\lambda)^{-1}\big)C(A-\lambda)^{-1}\\&=(D-\lambda)^{-1}C(A-\lambda)^{-1}+\overline{S_2(\lambda)}^{\,-1}\big(D-\lambda-\overline{S_2(\lambda)}\big)(D-\lambda)^{-1}C(A-\lambda)^{-1}\\&=(D-\lambda)^{-1}C(A-\lambda)^{-1}+\overline{S_2(\lambda)}^{\,-1}\,\overline{C(A-\lambda)^{-1}B}\,(D-\lambda)^{-1}C(A-\lambda)^{-1}\end{aligned}$$

which is compact by the second condition in i) and the first condition in ii). Analogously, one can show that the right upper corner is compact using the second condition in ii). This and the fact that $C(A-\lambda)^{-1}$ is bounded because of $\mathcal{D}(A)\subset\mathcal{D}(C)$ imply that also the left upper corner is compact. For the right lower corner, (2.4.8) and the resolvent identity for $A$ show that

$$\begin{aligned}&\overline{S_2(\lambda)}^{\,-1}-\big(\overline{S_2(\mu_0)}+\mu_0-\lambda\big)^{-1}\\&=\overline{S_2(\lambda)}^{\,-1}\big(\overline{S_2(\mu_0)}+\mu_0-\lambda-\overline{S_2(\lambda)}\big)\big(\overline{S_2(\mu_0)}+\mu_0-\lambda\big)^{-1}\\&=\overline{S_2(\lambda)}^{\,-1}\big(\overline{C(A-\lambda)^{-1}B}-\overline{C(A-\mu_0)^{-1}B}\big)\big(\overline{S_2(\mu_0)}+\mu_0-\lambda\big)^{-1}\\&=(\lambda-\mu_0)\,\overline{S_2(\lambda)}^{\,-1}C(A-\lambda)^{-1}\overline{(A-\mu_0)^{-1}B}\,\big(\overline{S_2(\mu_0)}+\mu_0-\lambda\big)^{-1}\end{aligned}$$

which is compact since it is the product of bounded operators and the compact operator $\overline{S_2(\lambda)}^{\,-1}C(A-\lambda)^{-1}$. □

The following analogue of Theorem 2.4.8 in terms of the first Schur complement can be obtained by comparing the resolvent of $\mathcal{A}$ with the resolvent of the block diagonal matrix (see (2.4.7))

$$\overline{\mathcal{A}}_{2,\mu_0}:=\begin{pmatrix}\overline{A-B(D-\mu_0)^{-1}C} & 0\\ 0 & D\end{pmatrix}=\begin{pmatrix}\overline{S_1(\mu_0)}+\mu_0 & 0\\ 0 & D\end{pmatrix},$$

where $\mu_0\in\rho(D)$ is arbitrary, but fixed.

**Theorem 2.4.9** *Let $\mathcal{D}(D)\subset\mathcal{D}(B)$, $\rho(A)\cap\rho(D)\neq\emptyset$, and let $\mathcal{D}(A)\cap\mathcal{D}(C)$ be a core of $A$. If, for some (and hence for all) $\mu\in\rho(A)\cap\rho(D)$,*

i) *$(D-\mu)^{-1}C$ and $B(D-\mu)^{-1}C$ are bounded on $\mathcal{D}(C)$,*
ii) *$(A-\mu)^{-1}B(D-\mu)^{-1}$ and $(D-\mu)^{-1}C(A-\mu)^{-1}$ are compact,*

*then, for every $\mu_0\in\rho(D)$ with $\rho\big(\overline{\mathcal{A}}\big)\cap\rho(D)\cap\rho\big(A-\overline{B(D-\mu_0)^{-1}C}\big)\neq\emptyset$, the difference of the resolvents $(\overline{\mathcal{A}}-\lambda)^{-1}$ and $(\overline{\mathcal{A}}_{2,\mu_0}-\lambda)^{-1}$ is compact for $\lambda\in\rho\big(\overline{\mathcal{A}}\big)\cap\rho(D)\cap\rho\big(A-\overline{B(D-\mu_0)^{-1}C}\big)$; in particular,*

$$\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big)=\sigma_{\mathrm{ess}}(D)\cup\sigma_{\mathrm{ess}}\big(A-\overline{B(D-\mu_0)^{-1}C}\big).$$

**Remark 2.4.10** Theorem 2.4.8 was first proved in [ALMS94, Theorem 2.2] in the special case that the block operator matrix is upper dominant, $A$ has compact resolvent, and $\overline{C(A-\mu)^{-2}B}$ is compact for some $\mu\in\rho(A)$, and, shortly after, in [Shk95, Theorem 2] for closable block operator matrices with $\mathcal{D}(A)\subset\mathcal{D}(C)$ under the slightly more general assumption that $C(A-\mu)^{-1}$ and $\overline{(A-\mu)^{-1}B}$ are compact for some $\mu\in\rho(A)$ (note that, if either $C$ or $B$ is bijective, this again implies that $A$ has compact resolvent). In both cases, it was shown that the difference of the corresponding block operator matrices $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}}_{1,\mu_0}$ is compact.

**Example 2.4.11** Theorem 2.4.8 applies to various matrix differential operators. Let, for example, $\Omega\subset\mathbb{R}^n$ be a domain and consider

$$\mathcal{A}=\begin{pmatrix}-\Delta+q & -\nabla\, b+w_1\\ c\,\mathrm{div}+w_2 & u\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\tag{2.4.9}$$

in $L_2(\Omega)\oplus L_2(\Omega)$. For simplicity, we assume the coefficients $q:\Omega\to M_n(\mathbb{C})$ and $b$, $c$, $w_1$, $w_2$, $u:\Omega\to\mathbb{C}$ to be sufficiently smooth. Note that if $b$, $c\not\equiv 0$, then the order of $A$ equals the sum of the orders of $B$ and $C$; otherwise the order of $A$ is strictly greater. In the particular case $b\equiv c\equiv 1$ and $q\equiv w_1\equiv w_2\equiv u\equiv 0$, the operator (2.4.9) is related to the so-called linearized Navier-Stokes operator or Stokes operator (see *e.g.* [LL87]).

First we consider bounded $\Omega\subset\mathbb{R}^n$. If $n=1$ and Dirichlet boundary conditions are imposed in the first component, then $A=-\mathrm{D}^2+q$ is a Sturm-Liouville operator with compact resolvent and thus $\sigma_{\mathrm{ess}}(A)=\emptyset$. It is easy to check that conditions i) and ii) of Theorem 2.4.8 are satisfied. If we choose $\mu_0\in\rho(A)$ arbitrary, *e.g.* $\mu_0=0$ if $q>0$, then Theorem 2.4.8 yields

$$\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big)=\sigma_{\mathrm{ess}}\big(D-\overline{C(A-\mu_0)^{-1}B}\big)=\sigma_{\mathrm{ess}}\,(u-b\,c)=(u-b\,c)\,\big(\overline{\Omega}\big);$$

in particular, if $b\equiv c\equiv 0$, this reproves the result of Example 2.4.3. Higher order ordinary matrix differential operators in $L_2(\Omega)^k\oplus L_2(\Omega)^l$ were treated in [ALMS94, Theorem 4.5]. For $n>1$, an analogous result (with $\sigma_{\mathrm{ess}}(A)\neq\emptyset$) may be found in [Kon02, Theorem 2].

Now let $\Omega \subset \mathbb{R}^n$ be unbounded. The case $n=1$, $I=[0,\infty)$ (or $\mathbb{R}$) was considered as an example in [Shk95]; there it was proved that if $q$, $b$, $c$, $w_1$, and $w_2$ tend to 0 for $|x|\to\infty$ (so that $\sigma_{\mathrm{ess}}(A)=[0,\infty)$), then

$$\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big) = [0,\infty) \cup \sigma_{\mathrm{ess}}\,(u-b\,c) = [0,\infty) \cup \overline{(u-b\,c)\,(\Omega)};$$

the conditions on the coefficients were weakened in [KM02, Proposition 4.1] for the self-adjoint case $c=\overline{b}$, $w_2=\overline{w_1}$ where it was assumed that $u$ is bounded from below and $q$ as well as $\big(|b|^2+|w_1|^2\big)/(|u|+1)$ tend to 0 for $|x|\to\infty$. Similar conditions were used in the case $n>1$ for $\Omega=\mathbb{R}^n$ in the previous papers [Kon97, Proposition 2], [Kon98, Theorem 2]. The case of higher order self-adjoint partial matrix differential operators for which the order of $A=(-\Delta)^l+q$ is $2l$ and the order of $B$ (and hence of $C=B^*$) is strictly less than $l$ was studied in [Kon97, Proposition 1]; then, under certain assumptions on the coefficients, $\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big)=[0,\infty)\cup\overline{u(\mathbb{R}^n)}$ (compare [Lut04, Theorem 6] for the case $n=1$ and multiplication operators $B$, $C$).

**Remark 2.4.12** Matrix differential operators (2.4.9) with $b\equiv c\equiv 0$ in $L_2(\mathbb{R})\oplus L_2(\mathbb{R})$ with periodic boundary conditions were studied in [HSV00], [HSV02]. As in the classical case of periodic Sturm-Liouville operators, the essential spectrum of $\mathcal{A}$ is the union of the spectra of all matrix differential operators $\mathcal{A}_\theta$, $\theta\in[0,2\pi)$, that arise when $\mathcal{A}$ is considered in $L_2(0,1)\oplus L_2(0,1)$ with boundary conditions $y_1(1)=\mathrm{e}^{\mathrm{i}\theta}y_1(0)$, $y_1'(1)=\mathrm{e}^{\mathrm{i}\theta}y_1'(0)$ in the first component. The above example shows that $\sigma_{\mathrm{ess}}(\mathcal{A}_\theta)=u([0,1])=u(\mathbb{R})$. Together with results on the discrete spectrum of $\mathcal{A}_\theta$, it was shown that $\sigma_{\mathrm{ess}}(\mathcal{A})$ consists of two sequences of bands, an infinite sequence to the right of $u(\mathbb{R})$ tending to $\infty$ and a finite or infinite sequence accumulating from the left at the lower end point of $u(\mathbb{R})$ (see [HSV00, Theorem 1]).

For self-adjoint block operator matrices, the assumptions of Theorem 2.4.8 were considerably weakened in [Kon98, Theorem 1] by requiring that only the difference of the *resolvents* of $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}}_{1,\mu_0}$ is compact. In the following we formulate this result as a special case of Theorem 2.4.8.

**Corollary 2.4.13** *Suppose that $A$ and $D$ are self-adjoint, $C=B^*$, and $\mathcal{D}(B)\cap\mathcal{D}(D)$ is a core of $D$. If*

i′) $\mathcal{D}(|A|^{1/2})\subset\mathcal{D}(B^*)$,
ii′) $(D-\mu)^{-1}B^*(A-\mu)^{-1}$ *is compact for some* $\mu\in\mathbb{C}\setminus\mathbb{R}$,

*then, for every $\mu_0\in\rho(A)$, the difference of the resolvents $\big(\overline{\mathcal{A}}-\lambda\big)^{-1}$ and*

$\big(\overline{\mathcal{A}}_{1,\mu_0} - \lambda\big)^{-1}$ *is compact for* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*; in particular,*

$$\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{ess}}(A) \cup \sigma_{\mathrm{ess}}\big(D - \overline{B^*(A-\mu_0)^{-1}B}\big).$$

***Proof.*** By the assumptions, $\mathbb{C} \setminus \mathbb{R} \subset \rho(A) \cap \rho(D)$ and $\mathcal{A}$ is essentially self-adjoint by Proposition 2.3.6. In the proof of the latter, it has also been shown that assumption i′) implies assumption i) of Theorem 2.4.8. Assumption ii′) implies assumption ii) of Theorem 2.4.8 because $(A-\mu)^{-1}B(D-\mu)^{-1} \subset \big((D-\overline{\mu})^{-1}B^*(A-\overline{\mu})^{-1}\big)^*$ for $\mu \in \mathbb{C}\setminus\mathbb{R}$. Moreover, $D - \overline{B^*(A-\mu_0)^{-1}B}$ is self-adjoint, being a bounded perturbation of the self-adjoint operator $D$. Hence $\mathbb{C}\setminus\mathbb{R} \subset \rho\big(\overline{\mathcal{A}}\big) \cap \rho(A) \cap \rho\big(D - \overline{C(A-\mu_0)^{-1}B}\big)$. Now the claims follow from Theorem 2.4.8. □

**Remark 2.4.14** In [AK05, Theorem 4.3] the above result was further extended to show that if, in addition to the assumptions i′) and ii′), one supposes that $(D-\mu)^{-1}B^*(A-\mu)^{-1}$ is a Hilbert-Schmidt operator, then the local wave operators $W_\Delta^\pm(\overline{\mathcal{A}}, \mathcal{A}_{\mu_0})$ exist and are complete, and for the absolutely continuous spectrum $\sigma_{\mathrm{ac}}\big(\overline{\mathcal{A}}\big)$ of $\overline{\mathcal{A}}$ one has

$$\sigma_{\mathrm{ac}}\big(\overline{\mathcal{A}}\big) \cap \Delta = \big(\sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{ac}}\big(D - \overline{B^*(A-\mu_0)^{-1}B}\big)\big) \cap \Delta$$

where $\Delta := \mathbb{R} \setminus \big(\sigma_{\mathrm{ess}}(A) \cup \sigma_{\mathrm{ess}}\big(D - \overline{B^*(A-\mu_0)^{-1}B}\big)\big)$ (see [Kat95, Section X.4] for the corresponding definitions). In order to prove this, it was shown that the difference of the corresponding resolvents is of trace class and hence a resolvent version of the Kato-Rosenblum theorem (see [Kat95, Theorem 4.12]) applies; the special case $b \equiv c \equiv 0$, $w_1 = w_2 = \sqrt{w}$ of Example 2.4.11 was treated before in [ALL01, Proposition 2.1, Corollary 2.1]. For matrix differential operators (2.4.9) in $L_2(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n)$, more results on the absence of singular continuous spectrum and eigenvalue accumulation were proved in [Kon96], [Kon98] using Mourre's commutator method.

In order to simplify the calculation of the essential spectrum of the Schur complement in Theorem 2.4.8, the following theorem is useful. Under certain relative compactness assumptions, it allows us to split off terms of the entries of the block operator matrix.

**Theorem 2.4.15** *Let $A$ be self-adjoint, $A \geq \alpha > 0$, $\rho(A) \cap \rho(D) \neq \emptyset$, let $\mathcal{D}(B) \cap \mathcal{D}(D)$ be a core of $D$, and let $\theta \in [0,1]$ be such that*

$$\mathcal{D}\big(A^\theta\big) \subset \mathcal{D}(C), \quad \mathcal{D}\big(A^{1-\theta}\big) \subset \mathcal{D}(B^*). \tag{2.4.10}$$

*Further, suppose that*

$$A = A_0 + A_1, \quad B = B_0 + B_1, \quad C = C_0 + C_1, \quad D = D_0 + D_1$$

*are such that*

i) $A_1$ *is* $A_0$*-compact and* $A_0 \geq \alpha_0 > 0$,
ii) $B_1$ *is* $B_0$*-compact,* $B_1^*$ *is* $B_0^*$*-compact, and* $B_0$ *is Fredholm,*
iii) $C_1$ *is* $C_0$*-compact,* $D_1$ *is* $D_0$*-compact,*
iv) $(A_0-\mu_0)^{-\theta}A_1(A_0-\mu_0)^{-(1-\theta)}$ *is compact for some* $\mu_0 \in \rho(A)\cap\rho(A_0)$.

*Then*

$$\sigma_{\mathrm{ess}}\big(D-\overline{C(A-\mu_0)^{-1}B}\big) = \sigma_{\mathrm{ess}}\big(D_0-\overline{C_0(A_0-\mu_0)^{-1}B_0}\big).$$

***Proof.*** Assumption (2.4.10) implies that the operators $C(A-\mu_0)^{-\theta}$, $B^*(A-\overline{\mu_0})^{-(1-\theta)}$ are bounded and hence so is the operator $C(A-\mu_0)^{-1}B = C(A-\mu_0)^{-\theta}(A-\mu_0)^{-(1-\theta)}B$ and its closure $\overline{C(A-\mu_0)^{-1}B}$. In order to see that $C_0(A_0-\mu_0)^{-1}B_0$ is bounded on $\mathcal{D}(B_0)$, we write

$$C_0(A_0-\mu_0)^{-1}B_0 = C_0(A_0-\mu_0)^{-\theta}(A_0-\mu_0)^{-(1-\theta)}B_0$$

and observe that

$$\begin{aligned} C_0(A_0-\mu_0)^{-\theta} &= \big(C_0(A-\mu_0)^{-\theta}\big)\big((A-\mu_0)^{\theta}(A_0-\mu_0)^{-\theta}\big),\\ B_0^*(A_0-\mu_0)^{-(1-\theta)} &= \big(B_0^*(A-\mu_0)^{-(1-\theta)}\big)\big((A-\mu_0)^{1-\theta}(A_0-\mu_0)^{-(1-\theta)}\big) \end{aligned}$$

are bounded. In fact, here the first factors are bounded because of (2.4.10) and $\mathcal{D}(B_0)=\mathcal{D}(B)$, $\mathcal{D}(C_0)=\mathcal{D}(C)$; the second factors are bounded since $A$ and $A_0$ are uniformly positive self-adjoint operators with $\mathcal{D}(A)=\mathcal{D}(A_0)$ and thus $\mathcal{D}(A_0^\theta)=\mathcal{D}(A^\theta)$ by Heinz' inequality (see [Kre71, Theorem 7.1]).

The claim follows from Theorem 2.1.13 if we show that the difference

$$\begin{aligned} &D-\overline{C(A-\mu_0)^{-1}B}-\big(D_0-\overline{C_0(A_0-\mu_0)^{-1}B_0}\big) \qquad (2.4.11)\\ &= D_1+\overline{C_0(A_0-\mu_0)^{-1}A_1(A-\mu_0)^{-1}B_0}-\overline{C_1(A-\mu_0)^{-1}B}-\overline{C_0(A-\mu_0)^{-1}B_1} \end{aligned}$$

is $\big(D-\overline{C(A-\mu_0)^{-1}B}\big)$-compact. Since $\overline{C(A-\mu_0)^{-1}B}$ is bounded, it is sufficient to prove that all four terms on the right hand side of (2.4.11) are $D$-compact. The operator $D_1$ is $D$-compact by assumption iii) and Lemma 2.1.8 ii). The third term is compact since

$$C_1(A-\mu_0)^{-1}B = C_1(A-\mu_0)^{-\theta}(A-\mu_0)^{-(1-\theta)}B,$$

$C_1(A-\mu_0)^{-\theta}$, $(A-\mu_0)^{-(1-\theta)}B$ are bounded due to the inclusions (2.4.10) and $\mathcal{D}(C)\subset\mathcal{D}(C_1)$, and $C_1(A-\mu_0)^{-\theta}$ is compact by assumption iii) and Lemma 2.1.8 ii). The fourth term is compact since so is its adjoint

$$\big(C_0(A-\mu_0)^{-1}B_1\big)^* = \big((A-\mu_0)^{-(1-\theta)}B_1\big)^*\big(C_0(A-\mu_0)^{-\theta}\big)^*;$$

to see this we note that $C_0(A-\mu_0)^{-\theta}$ is bounded because $\mathcal{D}(A^\theta)\subset\mathcal{D}(C)=\mathcal{D}(C_0)$ by (2.4.10) and assumption iii) and that

$$\big((A-\mu_0)^{-(1-\theta)}B_1\big)^* = B_1^*(A-\overline{\mu_0})^{-(1-\theta)}$$

is compact by (2.4.10), assumption ii), and Lemma 2.1.8 ii). In order to show that the second term is compact, we write

$$\begin{aligned} &C_0(A_0-\mu_0)^{-1}A_1(A-\mu_0)^{-1}B_0 \\ &= \big(C_0(A_0-\mu_0)^{-\theta}\big)\big((A_0-\mu_0)^{-(1-\theta)}A_1(A-\mu_0)^{-\theta}\big)\big((A-\mu_0)^{-(1-\theta)}B_0\big). \end{aligned}$$

It has been shown above that the first pair of factors on the right hand side is bounded. The middle group of factors is compact by assumption iv). The last pair of factors is bounded since

$$\big((A-\mu_0)^{-(1-\theta)}B_0\big)^* = B_0^*(A-\overline{\mu_0})^{-(1-\theta)}$$

and $B_0^*$ is $A^{1-\theta}$-bounded due to the second domain inclusion in (2.4.10), assumption ii), and Lemma 2.1.8 ii). □

Next we describe the essential spectrum of block operator matrices in terms of their quadratic complements.

**Theorem 2.4.16** *Let $\mathcal{D}(C) \subset \mathcal{D}(A)$ and let $C$ be boundedly invertible. If*

i) *$C^{-1}D$ is bounded on $\mathcal{D}(D)$,*

ii) *$T_2(\mu) = B-(A-\mu)C^{-1}(D-\mu)$ is closable for some (and thus all) $\mu \in \mathbb{C}$,*

*then*

$$\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{ess}}\big(\overline{T_2}\big).$$

***Proof.*** In the factorization (2.2.15) the first and the last factor are bounded and boundedly invertible and $C$ is bijective. Thus $\overline{\mathcal{A}}-\lambda$ is Fredholm if and only if so is $\overline{T_2(\lambda)}$ by [GGK90, Theorem XVII.3.1]. □

The following analogue of Theorem 2.4.16 for boundedly invertible $B$ can be proved in the same way; note that if $B$ and $C$ are invertible, Theorem 2.4.16 as well as Theorem 2.4.17 below may be applied.

**Theorem 2.4.17** *Let $\mathcal{D}(B) \subset \mathcal{D}(D)$ and let $B$ be boundedly invertible. If*

i) *$B^{-1}A$ is bounded on $\mathcal{D}(A)$,*

ii) *$T_1(\mu) = C-(D-\mu)B^{-1}(A-\mu)$ is closable for some (and thus all) $\mu \in \mathbb{C}$,*

*then*

$$\sigma_{\mathrm{ess}}\big(\overline{\mathcal{A}}\big) = \sigma_{\mathrm{ess}}\big(\overline{T_1}\big).$$

Applications of the above results are given in Chapter 3 to determine the essential spectra of block operator matrices arising in magnetohydrodynamics (see Theorem 3.1.3) and in fluid mechanics (see Proposition 3.2.2, Proposition 3.2.3, and Theorem 3.2.4).

## 2.5 Spectral inclusion

In this section we present a new method to localize the spectrum of unbounded block operator matrices. As in the bounded case, we introduce the concept of quadratic numerical range which refines the classical notion of the numerical range of a linear operator. In particular, we show that the quadratic numerical range has the spectral inclusion property for diagonally dominant and off-diagonally dominant block operator matrices of order 0.

The definition of the quadratic numerical range for bounded linear operators (see Definition 1.1.1) generalizes as follows to unbounded block operator matrices $\mathcal{A}$ of the form (2.2.1) with dense domain $\mathcal{D}(\mathcal{A})$,

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}_1 \oplus \mathcal{D}_2 = \big(\mathcal{D}(A) \cap \mathcal{D}(C)\big) \oplus \big(\mathcal{D}(B) \cap \mathcal{D}(D)\big).$$

**Definition 2.5.1** For $f \in \mathcal{D}_1$, $g \in \mathcal{D}_2$, $\|f\| = \|g\| = 1$, define the $2\times 2$-matrix

$$\mathcal{A}_{f,g} := \begin{pmatrix} (Af, f) & (Bg, f) \\ (Cf, g) & (Dg, g) \end{pmatrix} \in M_2(\mathbb{C}). \tag{2.5.1}$$

Then the set

$$W^2(\mathcal{A}) := \bigcup_{\substack{f \in \mathcal{D}_1, g \in \mathcal{D}_2, \\ \|f\| = \|g\| = 1}} \sigma_{\mathrm{p}}(\mathcal{A}_{f,g}) \tag{2.5.2}$$

is called the *quadratic numerical range* of the unbounded block operator matrix $\mathcal{A}$ (with respect to the block operator representation (2.2.1)).

**Remark 2.5.2** The equivalent descriptions of the quadratic numerical range in the bounded case given in Proposition 1.1.3 and Corollary 1.1.4 as well as the elementary properties stated in Proposition 1.1.7 generalize directly to the unbounded case; the only modification in the definitions, statements and/or proofs is that the conditions $f \in S_{\mathcal{H}_1}$ and $g \in S_{\mathcal{H}_2}$ have to be replaced by $f \in \mathcal{D}_1$, $\|f\| = 1$, and $g \in \mathcal{D}_2$, $\|g\| = 1$, respectively.

As in the bounded case, the quadratic numerical range of an unbounded block operator matrix consists of at most two components which need not be convex. However, in the unbounded case, both components may be unbounded subsets of the complex plane; *e.g.* if $A$ is an unbounded positive operator, $a := \inf W(A) > 0$, and

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix},$$

then $W^2(\mathcal{A}) = W(A) \cup W(-A) = (-\infty, -a] \cup [a, \infty)$.

The inclusion of the quadratic numerical range in the numerical range (see Theorem 1.1.8) and the inclusion of the numerical ranges of the diagonal elements in the quadratic numerical range (see Theorem 1.1.9) continue to hold for unbounded block operator matrices:

**Theorem 2.5.3** *For an unbounded block operator matrix $\mathcal{A}$, we have*

$$W^2(\mathcal{A}) \subset W(\mathcal{A}).$$

***Proof.*** The proof is completely analogous to the proof of Theorem 1.1.8 if we take $f \in \mathcal{D}_1$, $g \in \mathcal{D}_2$, $\|f\| = \|g\| = 1$. □

**Theorem 2.5.4** *Let $\mathcal{A}$ be an unbounded block operator matrix. Then*

i) $\dim \mathcal{H}_2 \geq 2 \implies W(A) \subset W^2(\mathcal{A})$,
ii) $\dim \mathcal{H}_1 \geq 2 \implies W(D) \subset W^2(\mathcal{A})$.

***Proof.*** We prove i); the proof of ii) is analogous. Let $\dim \mathcal{H}_2 \geq 2$. Since $\mathcal{D}_2$ is dense in $\mathcal{H}_2$, for each $f \in \mathcal{D}_1$, $\|f\| = 1$, there exists an element $g \in \mathcal{D}_2$, $\|g\| = 1$, such that $(Cf, g) = 0$. To see this, let $e_1$, $e_2 \in \mathcal{D}_2$, $\|e_1\| = \|e_2\| = 1$, be linearly independent. If $(Cf, e_1) = 0$ or $(Cf, e_2) = 0$, we can take $g = e_1$ or $g = e_2$, respectively. If $(Cf, e_1) \neq 0$ and $(Cf, e_2) \neq 0$, there exist non-zero constants $\alpha_1, \alpha_2 \in \mathbb{C}$ such that for $\widetilde{g} = \alpha_1 e_1 + \alpha_2 e_2$ we have $(Cf, \widetilde{g}) = \alpha_1 (Cf, e_1) + \alpha_2 (Cf, e_2) = 0$; in this case we choose $g := \widetilde{g}/\|\widetilde{g}\|$. With this choice of $g$ we obtain, as in the proof of Theorem 1.1.9,

$$\mathcal{A}_{f,g} = \begin{pmatrix} (Af, f) & (Bg, f) \\ 0 & (Dg, g) \end{pmatrix};$$

hence $(Af, f) \in W^2(\mathcal{A})$, and $W(A) \subset W^2(\mathcal{A})$ follows. □

The following results generalize Corollary 1.1.10 and Proposition 1.1.12 from the bounded case.

**Corollary 2.5.5** *Suppose that $\mathcal{A}$ is an unbounded block operator matrix and that $\dim \mathcal{H}_1 \geq 2$ and $\dim \mathcal{H}_2 \geq 2$.*

i) *If $W^2(\mathcal{A})$ consists of two disjoint components, $W^2(\mathcal{A}) = \mathcal{F}_1 \dot\cup \mathcal{F}_2$, they can be enumerated such that*

$$W(A) \subset \mathcal{F}_1, \quad W(D) \subset \mathcal{F}_2.$$

ii) *If $W(A) \cap W(D) \neq \emptyset$, then $W^2(\mathcal{A})$ consists of one component.*

***Proof.*** Assertions i) and ii) are immediate from Theorem 2.5.4 if we observe that $W(A)$ and $W(D)$ are convex and hence connected even if $A$ and $D$ are unbounded (see [Sto32, Theorem 4.7]). □

**Proposition 2.5.6** *If $\overline{W(A)} \cap \overline{W(D)} = \emptyset$, $B, C$ are bounded, and*

$$2\sqrt{\|B\|\,\|C\|} < \operatorname{dist}\big(W(A), W(D)\big),$$

*then $\overline{W^2(\mathcal{A})}$ consists of two disjoint components.*

***Proof.*** The proof is completely analogous to the proof of Proposition 1.1.12 if we observe that the convex sets $W(A)$ and $W(D)$ can be separated by a line having distance $\operatorname{dist}\big(W(A), W(D)\big)/2$ to both sets. □

To generalize the inclusion of the numerical ranges of the Schur complements (see Theorem 1.6.3) to the unbounded case, we restrict ourselves to the cases $\mathcal{D}(A) \subset \mathcal{D}(C)$ or $\mathcal{D}(D) \subset \mathcal{D}(B)$; otherwise, $S_1(\lambda)$ or $S_2(\lambda)$, respectively, need not have dense $\lambda$-independent domains (see Definition 2.2.12).

**Lemma 2.5.7** *For $f \in \mathcal{D}_1$, $g \in \mathcal{D}_2$, $f, g \neq 0$, and $\lambda \in \mathbb{C}$, we define*

$$\Delta(f, g; \lambda) := \det \begin{pmatrix} (Af, f) - \lambda(f, f) & (Bg, f) \\ (Cf, g) & (Dg, g) - \lambda(g, g) \end{pmatrix}.$$

i) *If $\mathcal{D}(D) \subset \mathcal{D}(B)$ and $f \in \mathcal{D}_1 = \mathcal{D}\big(S_1(\lambda)\big)$, $f \neq 0$, with $Cf \neq 0$, then*

$$\Delta\big(f, (D-\lambda)^{-1}Cf; \lambda\big) = \big(S_1(\lambda)f, f\big)\,\big(Cf, (D-\lambda)^{-1}Cf\big). \quad (2.5.3)$$

ii) *If $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $g \in \mathcal{D}_2 = \mathcal{D}\big(S_2(\lambda)\big)$, $g \neq 0$, with $Bg \neq 0$, then*

$$\Delta\big((A-\lambda)^{-1}Bg, g; \lambda\big) = \big(S_2(\lambda)g, g\big)\,\big(Bg, (A-\lambda)^{-1}Bg\big). \quad (2.5.4)$$

***Proof.*** The claims are immediate from the definition of $\Delta(f, g; \lambda)$ with $g := (D-\lambda)^{-1}Cf \neq 0$ and $f := (A-\lambda)^{-1}Bg \neq 0$, respectively, therein. □

**Theorem 2.5.8** *Let $\mathcal{A}$ be an unbounded block operator matrix.*

i) *If $\mathcal{D}(D) \subset \mathcal{D}(B)$, then $W(S_1) \subset W^2(\mathcal{A})$.*
ii) *If $\mathcal{D}(A) \subset \mathcal{D}(C)$, then $W(S_2) \subset W^2(\mathcal{A})$.*
iii) *If $\mathcal{A}$ is diagonally dominant, then $W(S_1) \cup W(S_2) \subset W^2(\mathcal{A})$.*

***Proof.*** Claim i) is obtained from formula (2.5.3) in the same way as in the proof of Theorem 1.6.3 for the bounded case; analogously, ii) follows from (2.5.4). Claim iii) is immediate from i) and ii). □

Next we establish the spectral inclusion property of the quadratic numerical range for unbounded block operator matrices (see Theorem 1.3.1 for the bounded case). The inclusion of the point spectrum holds without further assumptions:

**Theorem 2.5.9** *For an unbounded block operator matrix $\mathcal{A}$, we have*

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}).$$

***Proof.*** The inclusion can be proved in the same way as in the bounded case (see Theorem 1.3.1 and its proof), the only difference being that $(f\ g)^{\mathrm{t}} \in \mathcal{D}_1 \oplus \mathcal{D}_2$. □

The approximate point spectrum of an unbounded linear operator $T$ in a Hilbert space (see the definition in (1.3.4)) is contained in the closure of the numerical range, $\sigma_{\mathrm{app}}(T) \subset \overline{W(T)}$; in fact, for $\lambda \in \sigma_{\mathrm{app}}(T)$, there exists a sequence $(x_n)_1^\infty \subset \mathcal{D}(T)$, $\|x_n\| = 1$, with $(T-\lambda)x_n \to 0$, $n \to \infty$, and thus

$$\big|\big((T-\lambda)x_n, x_n\big)\big| \le \big\|(T-\lambda)x_n\big\| \longrightarrow 0, \quad n \to \infty.$$

We prove the analogue of this inclusion for the quadratic numerical range for diagonally dominant and for off-diagonally dominant block operator matrices of order 0 (see [Tre08, Theorem 4.2]).

**Theorem 2.5.10** *If $\mathcal{A}$ is diagonally dominant of order* 0*, then*

$$\sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$$

***Proof.*** Let $\lambda_0 \in \sigma_{\mathrm{app}}(\mathcal{A})$. Then there is a sequence $(\mathbf{f}_n)_1^\infty = \big((f_n\ g_n)^{\mathrm{t}}\big)_1^\infty \subset \mathcal{D}(A) \oplus \mathcal{D}(D)$, $\|f_n\|^2 + \|g_n\|^2 = 1$, such that $(\mathcal{A}-\lambda_0)\mathbf{f}_n \to 0$, $n \to \infty$, *i.e.*

$$\begin{aligned} (A-\lambda_0)f_n + Bg_n &=: h_n \longrightarrow 0, \\ Cf_n + (D-\lambda_0)g_n &=: k_n \longrightarrow 0, \end{aligned} \qquad n \to \infty. \tag{2.5.5}$$

Since the dominance order of $\mathcal{A}$ is 0 and hence $< 1$, the operator

$$\mathcal{S} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

is $\mathcal{A}$-bounded by Corollary 2.2.5. Thus $(\mathcal{A}-\lambda_0)\mathbf{f}_n \to 0$, $n \to \infty$, implies that $(\mathcal{S}\mathbf{f}_n)_1^\infty$ and hence also $\big(Bg_n\big)_1^\infty$ and $\big(Cf_n\big)_1^\infty$ are bounded. Then, by (2.5.5), $\big((A-\lambda_0)f_n\big)_1^\infty$ and $\big((D-\lambda_0)g_n\big)_1^\infty$ are bounded as well.

Now choose $\widehat{f}_n \in \mathcal{D}(A)$, $\widehat{g}_n \in \mathcal{D}(D)$ with $\|\widehat{f}_n\| = \|\widehat{g}_n\| = 1$, $f_n = \|f_n\|\widehat{f}_n$ and $g_n = \|g_n\|\widehat{g}_n$ for $n \in \mathbb{N}$, and consider

$$\Delta\big(\widehat{f}_n, \widehat{g}_n; \lambda\big) = \det \begin{pmatrix} (A\widehat{f}_n, \widehat{f}_n) - \lambda & (B\widehat{g}_n, \widehat{f}_n) \\ (C\widehat{f}_n, \widehat{g}_n) & (D\widehat{g}_n, \widehat{g}_n) - \lambda \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

Assume first that $\liminf_{n\to\infty} \|f_n\| > 0$ and $\liminf_{n\to\infty} \|g_n\| > 0$, without loss of generality $\|f_n\|, \|g_n\| > 0$, $n \in \mathbb{N}$. By (2.5.5), we have

$$\begin{aligned} (A\widehat{f}_n, \widehat{f}_n) - \lambda_0 &= \frac{1}{\|f_n\|}\Big((h_n, \widehat{f}_n) - \|g_n\|(B\widehat{g}_n, \widehat{f}_n)\Big), \\ (C\widehat{f}_n, \widehat{g}_n) &= \frac{1}{\|f_n\|}\Big((k_n, \widehat{g}_n) - \|g_n\|\big((D\widehat{g}_n, \widehat{g}_n) - \lambda_0\big)\Big), \end{aligned}$$

and thus

$$\Delta\big(\widehat{f}_n, \widehat{g}_n; \lambda_0\big) = \det \begin{pmatrix} \dfrac{1}{\|f_n\|}(h_n, \widehat{f}_n) & (B\widehat{g}_n, \widehat{f}_n) \\ \dfrac{1}{\|f_n\|}(k_n, \widehat{g}_n) & (D\widehat{g}_n, \widehat{g}_n) - \lambda_0 \end{pmatrix}. \tag{2.5.6}$$

The elements of the first column tend to 0 and the sequences with elements $B\widehat{g}_n = Bg_n/\|g_n\|$ and $(D-\lambda_0)\widehat{g}_n = (D-\lambda_0)g_n/\|g_n\|$ are bounded. Hence

$$\Delta\big(\widehat{f}_n, \widehat{g}_n; \lambda_0\big) \longrightarrow 0, \quad n \to \infty.$$

As $\Delta\big(\widehat{f}_n, \widehat{g}_n; \cdot\big)$ is a monic quadratic polynomial, we can write

$$\Delta\big(\widehat{f}_n, \widehat{g}_n; \lambda\big) = \big(\lambda - \lambda_n^1\big)\big(\lambda - \lambda_n^2\big), \quad n \in \mathbb{N}, \tag{2.5.7}$$

where $\lambda_n^1$, $\lambda_n^2$ are the solutions of the quadratic equation $\Delta\big(\widehat{f}_n, \widehat{g}_n; \lambda\big) = 0$ and hence $\lambda_n^1, \lambda_n^2 \in W^2(\mathcal{A})$. From (2.5.6) and (2.5.7) it follows that $\lambda_n^1 \to \lambda_0$ or $\lambda_n^2 \to \lambda_0$, $n \to \infty$, and thus $\lambda_0 \in \overline{W^2(\mathcal{A})}$.

Next we consider the case that $\liminf_{n\to\infty} \|g_n\| = 0$. Then we have $\liminf_{n\to\infty} \|f_n\| > 0$ since $\|f_n\|^2 + \|g_n\|^2 = 1$. Without loss of generality, we assume that $\lim_{n\to\infty} g_n = 0$ and $\|f_n\| > \gamma$, $n \in \mathbb{N}$, for some $\gamma \in (0,1]$.

First suppose that $\dim \mathcal{H}_2 \geq 2$. We show that $Bg_n \to 0$, $n \to \infty$. For this let $\varepsilon > 0$ be arbitrary. Since $\big((D-\lambda_0)g_n\big)_1^\infty$ is bounded, there exists an $M > 0$ such that $\|(D-\lambda_0)g_n\| < M$, $n \in \mathbb{N}$. Since $\mathcal{A}$ is diagonally dominant of order 0, the operator $B$ is $D$-bounded with $D$-bound 0 and hence there exist constants $a'_B$, $b_B \geq 0$ such that $b_B < \varepsilon/(2M)$ and

$$\|Bg_n\| \leq a'_B\|g_n\| + b_B\|(D-\lambda_0)g_n\|, \quad n \in \mathbb{N}.$$

If we choose $N \in \mathbb{N}$ such that $\|g_n\| < \varepsilon/(2a'_B)$, $n \geq N$, it follows that $\|Bg_n\| < \varepsilon$ for $n \geq N$. The fact that $Bg_n \to 0$, $n \to \infty$, and the first relation in (2.5.5) show that $(A-\lambda_0)f_n \to 0$, $n \to \infty$ and so $\lambda_0 \in \sigma_{\mathrm{app}}(A)$. Together with Theorem 2.5.4, we obtain

$$\lambda_0 \in \sigma_{\mathrm{app}}(A) \subset \sigma(A) \subset \overline{W(A)} \subset \overline{W^2(\mathcal{A})}.$$

If $\dim \mathcal{H}_2 = 1$, $B$ and $D$ are bounded operators. Then $g_n \to 0$ implies that $Bg_n \to 0$, $(D-\lambda_0)g_n \to 0$, $n \to \infty$, and, with $\widehat{f}_n$, $\widehat{g}_n$ chosen as above, $(A-\lambda_0)\widehat{f}_n \to 0$, $C\widehat{f}_n \to 0$, $n \to \infty$, by (2.5.5) and because we have $\big\|\widehat{f}_n\big\|^{-1} \leq \gamma^{-1}$, $n \in \mathbb{N}$. The boundedness of $B$ and $D$ also implies that $(B\widehat{g}_n)_1^\infty$, $\big((D-\lambda_0)\widehat{g}_n\big)_1^\infty$ are bounded. Therefore

$$\Delta\big(\widehat{f}_n, \widehat{g}_n; \lambda_0\big) = \det \begin{pmatrix} (A\widehat{f}_n, \widehat{f}_n) - \lambda_0 & (B\widehat{g}_n, \widehat{f}_n) \\ (C\widehat{f}_n, \widehat{g}_n) & (D\widehat{g}_n, \widehat{g}_n) - \lambda_0 \end{pmatrix} \longrightarrow 0, \quad n \to \infty.$$

Now, in the same way as above, it follows that $\lambda_0 \in \overline{W^2(\mathcal{A})}$.

The case $\liminf_{n\to\infty}\|f_n\|=0$ is treated analogously if we use that $C$ is $A$-bounded with $A$-bound 0. □

The next corollary is a direct consequence of Theorem 2.5.10 if we note that boundedness and relative compactness both imply relative boundedness with relative bound 0 (see [EE87, Corollary III.7.7], Proposition 2.1.7).

**Corollary 2.5.11** *If $B$ is bounded or $D$-compact and if $C$ is bounded or $A$-compact, then*

$$\sigma_{\rm app}(\mathcal{A})\subset\overline{W^2(\mathcal{A})}.$$

Theorem 2.5.10 and Corollary 2.5.11 generalize [LT98, Theorem 2.1] where it was assumed that $A$ and $D$ are closed and $B$ and $C$ are bounded and so, in particular, $\mathcal{A}$ is closed. In this case, by the closed graph theorem, the approximate point spectrum $\sigma_{\rm app}(\mathcal{A})$ is the complement of the set $r(\mathcal{A})$ of *points of regular type* which is defined as

$$r(\mathcal{A}):=\bigl\{\lambda\in\mathbb{C}:\exists\,C_\lambda>0\;\|(\mathcal{A}-\lambda)x\|\geq C_\lambda\|x\|,\;x\in\mathcal{D}(T)\bigr\}.$$

The inclusion of the approximate point spectrum for off-diagonally dominant block operator matrices was considered only recently (see [Tre08, Theorem 4.4]).

**Theorem 2.5.12** *If $\mathcal{A}$ is off-diagonally dominant of order* 0 *and $B$, $C$ are boundedly invertible, then*

$$\sigma_{\rm app}(\mathcal{A})\subset\overline{W^2(\mathcal{A})}.$$

***Proof.*** The first part of the proof is analogous to the proof of Theorem 2.5.10, now with $f_n,\widehat{f}_n\in\mathcal{D}(C)$ and $g_n,\widehat{g}_n\in\mathcal{D}(B)$; we continue to use the same notation. In fact, by Corollary 2.2.5, the operator

$$\mathcal{T}=\begin{pmatrix}A&0\\0&D\end{pmatrix}$$

is $\mathcal{A}$-bounded. Thus, in the same way as in the proof of Theorem 2.5.10, we can show that $\lambda_0\in\sigma_{\rm app}(\mathcal{A})$ implies that all sequences $\bigl((A-\lambda_0)f_n\bigr)_1^\infty$, $\bigl(Bg_n\bigr)_1^\infty$, $\bigl(Cf_n\bigr)_1^\infty$, $\bigl((D-\lambda_0)g_n\bigr)_1^\infty$ are bounded, $\Delta\bigl(\widehat{f}_n,\widehat{g}_n;\lambda_0\bigr)\to 0$, $n\to\infty$, and hence $\lambda_0\in\overline{W^2(\mathcal{A})}$ if $\liminf_{n\to\infty}\|f_n\|>0$ and $\liminf_{n\to\infty}\|g_n\|>0$.

It remains to consider the case $\liminf_{n\to\infty}\|g_n\|=0$, without loss of generality $\lim_{n\to\infty}g_n=0$ and $\|f_n\|>\gamma$, $n\in\mathbb{N}$, with some $\gamma\in(0,1]$; the case $\liminf_{n\to\infty}\|f_n\|=0$ is analogous.

If $\dim\mathcal{H}_2=1$, the proof is the same as the respective part of the proof of Theorem 2.5.10. If $\dim\mathcal{H}_2\geq 2$, we prove that $(Dg_n)_1^\infty$ tends to 0; since $C$ is boundedly invertible, this and the second relation in (2.5.5) yield

$$f_n = -C^{-1}(D - \lambda_0)g_n + C^{-1}k_n \longrightarrow 0, \quad n \to \infty,$$

a contradiction to $\|f_n\| > \gamma > 0$, $n \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary. Since $\big(Bg_n\big)_1^\infty$ is bounded, there exists an $M > 0$ such that $\|Bg_n\| < M$, $n \in \mathbb{N}$. Since $\mathcal{A}$ is off-diagonally dominant of order 0, the operator $D$ is $B$-bounded with $B$-bound 0 and hence there exist constants $a_D$, $b_D \geq 0$ such that $b_D < \varepsilon/(2M)$ and

$$\|Dg_n\| \leq a_D\|g_n\| + b_D\|Bg_n\|, \quad n \in \mathbb{N}.$$

If we choose $N \in \mathbb{N}$ such that $\|g_n\| < \varepsilon/(2a_D)$, $n \geq N$, it follows that $\|Dg_n\| < \varepsilon$ for $n \geq N$. □

**Corollary 2.5.13** *If $A$ is bounded or $C$-compact, $D$ is bounded or $B$-compact, and if $B$, $C$ are boundedly invertible, then*

$$\sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$$

The following example shows that Theorem 2.5.12 and Corollary 2.5.13 do not hold without the assumption that $B$ and $C$ are boundedly invertible.

**Example 2.5.14** Let $A = C = D = 0$ and let $B$ be a bijective (and hence closed) linear operator with dense domain $\mathcal{D}(B) \subsetneqq \mathcal{H}_2$. Then the block operator matrix

$$\mathcal{A} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{H}_1 \oplus \mathcal{D}(B),$$

is closed and off-diagonally dominant with $W^2(\mathcal{A}) = \{0\}$. If $\lambda \in \mathbb{C} \setminus \{0\}$, then $\mathcal{A} - \lambda$ is injective, the range $R(\mathcal{A} - \lambda) = \mathcal{H}_1 \oplus \mathcal{D}(B) \subsetneqq \mathcal{H}_1 \oplus \mathcal{H}_2$ is dense and hence $\lambda_0 \in \sigma_{\mathrm{c}}(\mathcal{A}) \subset \sigma_{\mathrm{app}}(\mathcal{A})$; if $\lambda = 0$, then $\mathcal{A} - \lambda$ is not injective and hence $\lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A}) \subset \sigma_{\mathrm{app}}(\mathcal{A})$. Thus $\sigma_{\mathrm{app}}(\mathcal{A}) = \mathbb{C}$ is not contained in $\overline{W^2(\mathcal{A})} = \{0\}$.

For a closed linear operator $T$, the spectral inclusion $\sigma(T) \subset \overline{W(T)}$ in the numerical range holds if every component of $\mathbb{C} \setminus \overline{W(T)}$ contains a point $\mu \in \rho(T)$, or, equivalently, a point $\mu$ such that the range $R(T - \mu) = \mathcal{H}$ (see [Kat95, Theorem V.3.2]). The analogue for the quadratic numerical range is as follows.

**Theorem 2.5.15** *Let $\mathcal{A}$ be closed and either diagonally dominant of order 0 or off-diagonally dominant of order 0 with $B$, $C$ boundedly invertible in the latter case. If $\Omega$ is a component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A})}$ that contains a point $\mu \in \rho(\mathcal{A})$, then $\Omega \subset \rho(\mathcal{A})$; in particular, if every component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A})}$ contains a point $\mu \in \rho(\mathcal{A})$, then*

$$\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$$

***Proof.*** For $\lambda \in r(\mathcal{A})$, the range $R(\mathcal{A}-\lambda)$ is closed and the mapping $\lambda \mapsto \dim R(\mathcal{A}-\lambda)^{\perp}$ is constant on every component of $r(\mathcal{A})$ (see *e.g.* [Kat95, Theorem V.3.2]). Thus, by Theorem 2.5.10 or 2.5.12, respectively, the same is true on each component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A})} \subset \mathbb{C} \setminus \sigma_{\mathrm{app}}(\mathcal{A}) = r(\mathcal{A})$. By assumption, it follows that $R(\mathcal{A}-\lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \overline{W^2(\mathcal{A})}$, that is, $\mathbb{C} \setminus \overline{W^2(\mathcal{A})} \subset \rho(\mathcal{A})$. □

Since the closure $\overline{\mathcal{A}}$ of a closable block operator matrix $\mathcal{A}$ need not be a block operator matrix, the quadratic numerical range of $\overline{\mathcal{A}}$ is, in general, not defined. However, due to the next lemma, the closure $\overline{W^2(\mathcal{A})}$ of the quadratic numerical range still provides an enclosure for $\sigma_{\mathrm{p}}(\overline{\mathcal{A}})$ and $\sigma_{\mathrm{app}}(\overline{\mathcal{A}})$.

**Lemma 2.5.16** *Let $T$ be a closable linear operator in a Banach space $E$ with closure $\overline{T}$. Then*

$$\sigma_{\mathrm{p}}(\overline{T}) \subset \sigma_{\mathrm{app}}(T), \quad \sigma_{\mathrm{app}}(\overline{T}) = \sigma_{\mathrm{app}}(T).$$

***Proof.*** If $\lambda \in \sigma_{\mathrm{p}}(\overline{T})$, then there exists an $x \in \mathcal{D}(\overline{T})$, $x \neq 0$, such that $(\overline{T}-\lambda)x = 0$. By definition of the closure, there is a sequence $(x_n)_1^\infty \subset \mathcal{D}(T)$ with $x_n \to x$, without loss of generality $\|x_n\| \geq \gamma$ for some $\gamma > 0$, and $(T-\lambda)x_n \to (\overline{T}-\lambda)x = 0$, $n \to \infty$. This shows that $\lambda \in \sigma_{\mathrm{app}}(T)$.

The inclusion $\sigma_{\mathrm{app}}(T) \subset \sigma_{\mathrm{app}}(\overline{T})$ is obvious. Vice versa, if $\lambda \in \sigma_{\mathrm{app}}(\overline{T})$, then there exists a sequence $(x_n)_1^\infty \subset \mathcal{D}(\overline{T})$ with $\|x_n\| = 1$, $n \in \mathbb{N}$, and $(\overline{T}-\lambda)x_n \to 0$, $n \to \infty$. Let $\varepsilon > 0$ be arbitrary such that $\varepsilon < 3|\lambda|/2$. By definition of the closure, for every $n \in \mathbb{N}$ there exists an element $y_n \in \mathcal{D}(T)$ such that $\|y_n - x_n\| < \varepsilon/(3|\lambda|)$ and $\|Ty_n - \overline{T}x_n\| < \varepsilon/3$. Choose $n_0 \in \mathbb{N}$ so that $\|(\overline{T}-\lambda)x_n\| < \varepsilon/3$ for $n \geq n_0$. Then, for $n \geq n_0$,

$$\|(T-\lambda)y_n\| \leq \|Ty_n - \overline{T}x_n\| + |\lambda| \, \|y_n - x_n\| + \|(\overline{T}-\lambda)x_n\| < \varepsilon.$$

Since $\varepsilon < 3|\lambda|/2$, we have $\|y_n\| \geq \|x_n\| - \|x_n - y_n\| \geq 1/2$ for all $n \in \mathbb{N}$. This shows that $\lambda \in \sigma_{\mathrm{app}}(T)$. □

**Corollary 2.5.17** *Let $\mathcal{A}$ be a closable block operator matrix with closure $\overline{\mathcal{A}}$. If the quadratic numerical range of $\mathcal{A}$ satisfies the spectral inclusion $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A})$ and $\sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$, then*

$$\sigma_{\mathrm{p}}(\overline{\mathcal{A}}) \subset \overline{W^2(\mathcal{A})}, \quad \sigma_{\mathrm{app}}(\overline{\mathcal{A}}) \subset \overline{W^2(\mathcal{A})}.$$

In the following we consider unbounded block operator matrices with certain additional structures. First we study block operator matrices such that $C = B^*$ and $A$, $-D$ are uniformly accretive (see Theorem 1.2.1 for the bounded case), *i.e.* there are constants $\alpha$, $\delta > 0$ with

$$\operatorname{Re} W(D) \leq -\delta < 0 < \alpha \leq \operatorname{Re} W(A).$$

The next theorem shows that this gap between the diagonal entries $A$ and $D$ is retained as a spectral gap for the whole block operator matrix $\mathcal{A}$, even in the off-diagonally dominant case where $B^*$ and $B$ are stronger than $A$ and $D$, respectively (see [Tre08, Theorem 5.2]).

**Theorem 2.5.18** *Let $\mathcal{A}$ be a closable block operator matrix of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

*and define the sector* $\Sigma_\omega := \{ r\mathrm{e}^{\mathrm{i}\phi} : r \geq 0, |\phi| \leq \omega \}$ *for* $\omega \in [0, \pi)$. *If there exist* $\alpha$, $\delta > 0$ *and angles* $\varphi, \vartheta \in [0, \pi/2]$ *such that*

$$W(D) \subset \{ z \in -\Sigma_\varphi : \operatorname{Re} z \leq -\delta \}, \quad W(A) \subset \{ z \in \Sigma_\vartheta : \operatorname{Re} z \geq \alpha \},$$

*then, with* $\theta := \max\{\varphi, \vartheta\}$,

$$\sigma_\mathrm{p}(\overline{\mathcal{A}}) \subset \{ z \in -\Sigma_\theta : \operatorname{Re} z \leq -\delta \} \cup \{ z \in \Sigma_\theta : \operatorname{Re} z \geq \alpha \}.$$

*Suppose, in addition, that $\mathcal{A}$ is either diagonally dominant or off-diagonally dominant of order* 0 *and, in the latter case, $B$ is boundedly invertible. If there exists a point* $\mu \in \rho(A) \cap \rho(D) \cap \{ z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha \}$, *then $\mathcal{A}$ is closed and*

$$\sigma(\mathcal{A}) \subset \{ z \in -\Sigma_\theta : \operatorname{Re} z \leq -\delta \} \cup \{ z \in \Sigma_\theta : \operatorname{Re} z \geq \alpha \}.$$

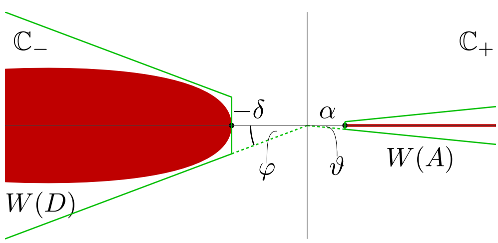



Figure 2.1 Assumptions on $A$ and $D$ in Theorem 2.5.18.

***Proof.*** By Lemma 1.2.2, the assumptions imply that

$$\overline{W^2(\mathcal{A})} \subset \{ z \in -\Sigma_\theta : \operatorname{Re} z \leq -\delta \} \cup \{ z \in \Sigma_\theta : \operatorname{Re} z \geq \alpha \} =: \Xi. \quad (2.5.8)$$

By Theorem 2.5.10 and Corollary 2.5.17, we have $\sigma_\mathrm{p}(\overline{\mathcal{A}}) \subset \overline{W^2(\mathcal{A})} \subset \Xi$. For the inclusion of the spectrum, we first note that in the off-diagonally dominant case, $B^*$ is boundedly invertible since so is $B$. That $\mathcal{A}$ is closed and $\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$ follows from Theorems 2.5.12, 2.5.15, and Corollary 2.5.17

if we show that $\mathbb{C} \setminus \Xi$ contains a point $\mu \in \rho(\mathcal{A})$; in fact, we will show that $\mathrm{i}\mathbb{R} \cap \rho(\mathcal{A}) \neq \emptyset$. To prove this, we consider the block operator matrices

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

Due to the assumptions on $A$, $D$, we have $\sigma(A) \subset \overline{W(A)}$, $\sigma(D) \subset \overline{W(D)}$ and

$$\big\|(\mathcal{T} - \mathrm{i}\mu)^{-1}\big\| \leq \frac{(\cos\theta)^{-1}}{|\mu|}, \quad \mu \in \mathbb{R} \setminus \{0\}.$$

If $B$ is boundedly invertible, it is closed and hence $\mathcal{S}$ is self-adjoint (see the subsequent Proposition 2.6.3) so that

$$\big\|(\mathcal{S} - \mathrm{i}\mu)^{-1}\big\| \leq \frac{1}{|\mu|}, \quad \mu \in \mathbb{R} \setminus \{0\}.$$

If $\mathcal{A}$ is diagonally dominant of order 0, then $\mathcal{S}$ is $\mathcal{T}$-bounded with $\mathcal{T}$-bound 0; if $\mathcal{A}$ is off-diagonally dominant of order 0, then $\mathcal{T}$ is $\mathcal{S}$-bounded with $\mathcal{S}$-bound 0 (see Proposition 2.2.5). In both cases, the assumptions of Corollary 2.1.5 are satisfied and so $\{\mathrm{i}\mu : |\mu| \geq R\} \subset \rho(\mathcal{T} + \mathcal{S}) = \rho(\mathcal{A})$ for some $R > 0$. □

**Remark 2.5.19** If $\mathcal{A}$ is self-adjoint, clearly, a point $\mu \in \rho(A) \cap \rho(D) \cap \{z \in \mathbb{C} : -\delta < \mathrm{Re}\, z < \alpha\}$ exists; in this case, $\sigma(\mathcal{A}) \subset \overline{\Lambda_-}(\mathcal{A}) \,\dot{\cup}\, \overline{\Lambda_+}(\mathcal{A}) \subset \mathbb{R}$ and

$$\overline{\Lambda_-}(\mathcal{A}) \subset \big\{z \in \mathbb{R} : \mathrm{Re}\, z \leq -\delta\big\}, \quad \overline{\Lambda_+}(\mathcal{A}) \subset \big\{z \in \mathbb{R} : \mathrm{Re}\, z \geq \alpha\big\}.$$

This spectral inclusion was first proved by J. Weidmann (see [Wei80, Theorem 7.25]) using the spectral theorem; in [AL95, Theorem 2.1], a shorter proof was given by showing that for $\lambda \in (-\delta, \alpha)$ the inverse of the first Schur complement $S_1(\lambda)$ and thus the resolvent $(\mathcal{A} - \lambda)^{-1}$ exist (see Theorem 2.3.3).

In the following example we use the above theorem to estimate the smallest eigenvalue in modulus of Dirac operators on certain compact manifolds with warped product metric (see [KT03]). Examples of such manifolds are manifolds with $S^1$-symmetry, in the simplest case ellipsoids. For a one-dimensional basis (like for an ellipsoid), such estimates may be obtained by Sturm-Liouville techniques (see *e.g.* [Kra00], [Kra01]); for higher dimensional bases, Theorem 2.5.18 provides an eigenvalue estimate that can be read off elegantly from the diagonal entries (see [KT03, Theorem 7.2]).

**Example 2.5.20** We consider the Dirac operator $D_{\mathcal{M}}$ on a closed Riemannian spin manifold $\mathcal{M}$ with warped product metric. These manifolds are complete Riemannian spin manifolds and hence $D_{\mathcal{M}}$ is an essentially self-adjoint operator acting on the space of spinors $\Gamma\Sigma_{\mathcal{M}}$, *i.e.* on sections of a certain $2^{[\frac{\dim \mathcal{M}}{2}]}$-dimensional complex vector bundle, the so-called spinor bundle $\Sigma_{\mathcal{M}} \to \mathcal{M}$. Since the manifold $\mathcal{M}$ is closed, the Dirac operator $D_{\mathcal{M}}$ has discrete spectrum. The kernel is not a topological but a conformal invariant and only few estimates are known for the first positive eigenvalue. Note that, if the dimension of $\mathcal{M}$ is even, then $\sigma(D_{\mathcal{M}})$ is symmetric to 0. For details on Dirac operators on manifolds we refer to [LM89] and [Fri00].

The manifold $\mathcal{M}$ with its warped product metric is defined as follows. Let $(\mathcal{B}^m, g_{\mathcal{B}})$, $(\mathcal{F}^k, g_{\mathcal{F}})$ be closed Riemannian spin manifolds of dimensions $m$ and $k$, respectively. For a positive $C^\infty$-function $f\colon \mathcal{B}^m \to \mathbb{R}^+$ we denote by $\mathcal{M} := \mathcal{B}^m \times_f \mathcal{F}^k := (\mathcal{B}^m \times \mathcal{F}^k, g_{\mathcal{B}} + f^2 g_{\mathcal{F}})$ the warped product of $\mathcal{B}^m$ and $\mathcal{F}^k$ with the product spin structure. In the following we always write $\mathcal{B}$, $\mathcal{F}$, and $\mathcal{B} \times_f \mathcal{F}$ to shorten the notation. To introduce the Dirac operator on $\mathcal{M}$, one has to distinguish the cases $m$ even, $m$ odd and $k$ even, and $m, k$ odd. For simplicity, we restrict ourselves to the case $m$ even; for the other cases, we refer the reader to [KLT04, Example 4.12].

For a manifold $\mathcal{X}$ we denote by $\Sigma_{\mathcal{X}}$ the spinor bundle over $\mathcal{X}$ and by $\pi_{\mathcal{X}} : \mathcal{M} \to \mathcal{X}$ the projection of $\mathcal{M}$ onto $\mathcal{X}$. Then, for the spinor bundles on $\mathcal{M}$, we have $\Sigma_{\mathcal{M}} \cong \pi_{\mathcal{B}}^* \Sigma_{\mathcal{B}} \otimes \pi_{\mathcal{F}}^* \Sigma_{\mathcal{F}}$. For an even-dimensional spin manifold $\mathcal{X}$, there is a natural splitting $\Sigma_{\mathcal{X}} = \Sigma_{\mathcal{X}}^+ \oplus \Sigma_{\mathcal{X}}^-$. With respect to this decomposition, the Dirac operator $D_{\mathcal{X}}$ which exchanges the positive and negative spinors has the form

$$D_{\mathcal{X}} = \begin{pmatrix} 0 & D_{\mathcal{X}}^+ \\ D_{\mathcal{X}}^- & 0 \end{pmatrix}.$$

The warped product structure of the manifold $\mathcal{M}$ allows us to write the Dirac operator $D_{\mathcal{M}}$ as a direct sum of off-diagonally dominant block operator matrices. To this end, we decompose the space of spinors over $\mathcal{M}$ along the eigenspaces of the Dirac operator on the fibre $\mathcal{F}$. More exactly, for every eigenvalue $\Lambda \in \sigma_{\rm p}(D_{\mathcal{F}})$, let $\mathcal{E}_\Lambda \to \mathcal{B}$ be the vector bundle with fibre $\mathcal{E}_{\Lambda,b} := E\big(\frac{\Lambda}{f(b)}, D_{f(b)\mathcal{F}}\big)$ trivialised by $(\frac{\pi_{\mathcal{F}}^* e_{\Lambda,1}}{f^{k/2}}, \dots, \frac{\pi_{\mathcal{F}}^* e_{\Lambda,r(\Lambda)}}{f^{k/2}})$ where $(e_{\Lambda,1}, \dots, e_{\Lambda,r(\Lambda)})$ is an orthonormal basis of the eigenspace $E(\Lambda, D_{\mathcal{F}})$ and $r(\Lambda)$ is the multiplicity of $\Lambda$. For $\Lambda \in \sigma_{\rm p}(D_{\mathcal{F}})$ we define $W_\Lambda := \Gamma_{\mathcal{B}}(\Sigma_{\mathcal{B}} \otimes \mathcal{E}_\Lambda) = \Gamma_{\mathcal{B}}\big(\Sigma_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}\big)$. Then the space of spinors decomposes as

$$\Gamma_{\mathcal{M}}(\Sigma_{\mathcal{M}}) = \bigoplus_{\Lambda \in \sigma_{\rm p}(D_{\mathcal{F}})} W_\Lambda;$$

a spinor $\Psi \in \Gamma_{\mathcal{M}}(\Sigma_{\mathcal{M}})$ is called a *spinor of weight* $\Lambda$ if $\Psi \in W_\Lambda$.

For the corresponding decomposition of the Dirac operator on $\mathcal{M}$, we fix $\Lambda \in \sigma_{\rm p}(D_{\mathcal{F}})$ and define the Hilbert spaces

$$\mathcal{H}_{1,\Lambda} := L^2\big(\Sigma^+_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}\big), \quad \mathcal{H}_{2,\Lambda} := L^2\big(\Sigma^-_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}\big), \quad \mathcal{H}_\Lambda := \mathcal{H}_{1,\Lambda} \oplus \mathcal{H}_{2,\Lambda}.$$

We introduce the bounded self-adjoint operators $A_{i,\Lambda}$ in $\mathcal{H}_{i,\Lambda}$, $i=1,2$, by

$$A_{i,\Lambda} : \mathcal{H}_{i,\Lambda} \to \mathcal{H}_{i,\Lambda}, \quad A_{i,\Lambda}\Psi_i = \frac{\Lambda}{f}\Psi_i,$$

and the closed operator $B_\Lambda$ from $\mathcal{H}_{2,\Lambda}$ into $\mathcal{H}_{1,\Lambda}$ by

$$\begin{aligned} \mathcal{D}(B_\Lambda) &= \big\{\Psi_2 \in \mathcal{H}_{2,\Lambda} \colon \Psi_2 \in W^{1,2}\big(\Sigma^-_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}\big)\big\}, \\ B_\Lambda &= D_{\mathcal{B}} = \begin{pmatrix} 0 & D^+_{\mathcal{B}} \\ D^-_{\mathcal{B}} & 0 \end{pmatrix}. \end{aligned} \tag{2.5.9}$$

Then the Dirac operator $D_{\mathcal{M}}$ on the manifold $\mathcal{M}$ can be written as

$$D_{\mathcal{M}} = \bigoplus_{\Lambda \in \sigma_{\mathrm{p}}(D_{\mathcal{F}})} \mathcal{A}_\Lambda, \quad \mathcal{A}_\Lambda := \begin{pmatrix} A_{1,\Lambda} & B_\Lambda \\ B^*_\Lambda & -A_{2,\Lambda} \end{pmatrix}; \tag{2.5.10}$$

an eigenvalue $\lambda$ of $D_{\mathcal{M}}$ is called an *eigenvalue of weight* $\Lambda$ if there is an eigenspinor $\Psi$ associated with $\lambda$ that belongs to $W_\Lambda$. Since $B_\Lambda$ is closed and $A_{1,\Lambda}$, $A_{2,\Lambda}$ are bounded, the block operator matrices $\mathcal{A}_\Lambda$ in $\mathcal{H}_\Lambda$ with domain $\mathcal{D}(\mathcal{A}_\Lambda) = W^{1,2}\big(\Sigma^+_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}\big) \oplus W^{1,2}\big(\Sigma^-_{\mathcal{B}} \otimes \mathbb{C}^{r(\Lambda)}\big)$ are off-diagonally dominant and self-adjoint (see the subsequent Theorem 2.6.6 iii)).

Suppose that $0 \notin \sigma_{\mathrm{p}}(\mathrm{D}_{\mathcal{F}})$. Then every operator $\mathcal{A}_\Lambda$ satisfies the assumptions of Theorem 2.5.18 since, for $\Psi_1 \in \mathcal{D}(A_{1,\Lambda})$,

$$\big(A_{1,\Lambda}\Psi_1, \Psi_1\big)_{L_2} = \Lambda \int_{\mathcal{B}} \left|\frac{\Psi_1}{f}\right|^2 \mathrm{d}\mathcal{B} \geq \frac{\Lambda}{f_{\max}} \|\Psi_1\|^2_{L_2},$$

and analogously for $A_{2,\Lambda}$. Hence, Theorem 2.5.18 yields the inclusions

$$\sigma_{\mathrm{p}}(\mathcal{A}_\Lambda) \subset \left(-\infty, -\frac{\Lambda}{f_{\max}}\right] \cup \left[\frac{\Lambda}{f_{\max}}, \infty\right) \tag{2.5.11}$$

and thus

$$\sigma_{\mathrm{p}}\big(D_{\mathcal{M}}\big) = \bigcup_{\Lambda \in \sigma_{\mathrm{p}}(D_{\mathcal{F}})} \sigma_{\mathrm{p}}(\mathcal{A}_\Lambda) \subset (-\infty, -\alpha] \cup [\alpha, \infty)$$

where

$$\alpha := \min_{\Lambda \in \sigma_{\mathrm{p}}(D_{\mathcal{F}})} \big\{|\lambda| : \lambda \in \sigma_{\mathrm{p}}(\mathcal{A}_\Lambda)\big\} \geq \min\left\{\frac{|\Lambda|}{f_{\max}} : \Lambda \in \sigma_{\mathrm{p}}(D_{\mathcal{F}})\right\}.$$

For the particular case that $\mathcal{F} = S^k$ is the $k$-dimensional sphere, the smallest eigenvalue in modulus of $D_{S^k}$ is $k/2$ (see [Sul80]) and so in this case

$$\sigma_{\mathrm{p}}\big(D_{\mathcal{M}}\big) \subset \left(-\infty, -\frac{k}{2f_{\max}}\right] \cup \left[\frac{k}{2f_{\max}}, \infty\right).$$

If the diagonal entries $A$ and $D$ are either both bounded from below or from above and $B$ is bounded, then the quadratic numerical range provides the following estimate for the lower or upper bound, respectively, of the block operator matrix $\mathcal{A}$ (see [Tre08, Theorems 5.5, 5.6] and compare Proposition 1.3.6 in the bounded case).

**Proposition 2.5.21** *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

*with $A = A^*$, $D = D^*$ either both semi-bounded from below or from above and bounded $B$. Then the spectrum of $\mathcal{A}$ satisfies the following estimates:*

i) *If $A$ and $D$ are bounded from below and*

$$\delta_B^- := \|B\| \tan\left(\frac{1}{2}\arctan\left(\frac{2\|B\|}{|\min\sigma(A) - \min\sigma(D)|}\right)\right),$$

*then $\mathcal{A}$ is bounded from below with*

$$\min\bigl\{\min\sigma(A), \min\sigma(D)\bigr\} - \delta_B^- \le \min\sigma(\mathcal{A}) \le \min\bigl\{\min\sigma(A), \min\sigma(D)\bigr\}.$$

ii) *If $A$ and $D$ are bounded from above and*

$$\delta_B^+ := \|B\| \tan\left(\frac{1}{2}\arctan\left(\frac{2\|B\|}{|\max\sigma(A) - \max\sigma(D)|}\right)\right),$$

*then $\mathcal{A}$ is bounded from above with*

$$\max\bigl\{\max\sigma(A), \max\sigma(D)\bigr\} \le \max\sigma(\mathcal{A}) \le \max\bigl\{\max\sigma(A), \max\sigma(D)\bigr\} + \delta_B^+.$$

***Proof.*** The proof of the fact that the quadratic numerical range satisfies the stated estimates is analogous to the proof of Proposition 1.3.6. In order to obtain the estimates of the spectrum, we use Corollary 2.5.11, Theorem 2.5.15, and the fact that $\mathbb{C}\setminus\mathbb{R} \subset \rho(\mathcal{A})$ since $\mathcal{A}$ is self-adjoint. □

Next we consider unbounded self-adjoint block operator matrices with diagonal entries $A$, $D$ having disjoint spectra and bounded $B$. In this case, an analogue of Proposition 1.3.7 and more advanced results on solutions of corresponding Riccati equations (see Section 2.10) were proved in [KMM05, Theorem 1, Remark 3.3] using factorization results of Virozub and Matsaev and the Daleckii–Krein formula. Here we obtain a simpler proof of the spectral inclusion by means of the quadratic numerical range.

**Proposition 2.5.22** *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

*with $A = A^*$, $D = D^*$, $\delta_{A,D} := \operatorname{dist}\bigl(\sigma(A), \sigma(D)\bigr) > 0$, and bounded $B$.*

*Define*

$$\delta_B := \|B\| \tan\left(\frac{1}{2}\arctan\left(\frac{2\|B\|}{\delta_{A,D}}\right)\right).$$

i) *Then*

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A)\,\dot\cup\,\sigma(D)\big) \le \delta_B\big\}.$$

ii) *If* $\|B\| < \frac{\sqrt{3}}{2}\,\delta_{A,D}$, *then* $\delta_B < \frac{\delta_{A,D}}{2}$ *and* $\sigma(\mathcal{A}) = \sigma_1 \,\dot\cup\, \sigma_2$ *with* $\sigma_1, \sigma_2 \neq \emptyset$ *and*

$$\sigma_1 \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A)\big) \le \delta_B\big\} \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A)\big) < \delta_{A,D}/2\big\},$$
$$\sigma_2 \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(D)\big) \le \delta_B\big\} \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(D)\big) < \delta_{A,D}/2\big\}.$$

iii) *If* $\operatorname{conv}\sigma(A) \cap \sigma(D) = \emptyset$ *and* $\|B\| < \sqrt{2}\,\delta_{A,D}$, *then* $\delta_B < \delta_{A,D}$ *and* $\sigma(\mathcal{A}) = \sigma_1 \,\dot\cup\, \sigma_2$ *with* $\sigma_1, \sigma_2 \neq \emptyset$ *and*

$$\sigma_1 \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A)\big) \le \delta_B\big\} \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A)\big) < \delta_{A,D}\big\},$$
$$\sigma_2 \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A)\big) \ge \delta_{A,D}\big\}.$$

***Proof.*** Since $\mathcal{A}$ is self-adjoint and hence $\mathbb{C} \setminus \mathbb{R} \subset \rho(\mathcal{A})$, the inclusion $\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$ holds by Corollary 2.5.11 and Theorem 2.5.15. That $\overline{W^2(\mathcal{A})}$ satisfies the estimates claimed for the spectrum can be proved in the same way as Proposition 1.3.7 if we use Proposition 2.5.21. □

## 2.6 Symmetric and $\mathcal{J}$-symmetric block operator matrices

In this section we consider block operator matrices $\mathcal{A}$ with real quadratic numerical range. Like in the bounded case, the condition that $W^2(\mathcal{A}) \subset \mathbb{R}$ does not imply symmetry of the block operator matrix, it only implies symmetry with respect to a possibly indefinite inner product $[\cdot,\cdot] = (\mathcal{J}\cdot,\cdot)$ on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where

$$\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(see Theorem 1.1.15 for the bounded case). For unbounded operators, however, there is a difference between $\mathcal{J}$-symmetric and $\mathcal{J}$-self-adjoint operators, depending on whether $\mathcal{J}\mathcal{A}$ is only symmetric or self-adjoint in the Hilbert space $\mathcal{H}$ (see Definition 1.1.14). In particular, the relation

$$[\mathcal{A}\mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathcal{A}\mathbf{y}], \quad \mathbf{x}, \mathbf{y} \in \mathcal{D}(\mathcal{A}),$$

only implies that $\mathcal{A}$ is $\mathcal{J}$-symmetric. Note that, for a $\mathcal{J}$-self-adjoint operator $\mathcal{A}$, the spectrum $\sigma(\mathcal{A})$ as well as the resolvent set $\rho(\mathcal{A})$ may be empty.

For details on unbounded operators in spaces with indefinite inner products we refer to [Bog74] and, for a brief overview, to [LNT06, Section 2].

We begin by presenting some simple criteria for an unbounded block operator matrix $\mathcal{A}$ given by (2.2.1), (2.2.2) to be symmetric (self-adjoint) and $\mathcal{J}$-symmetric ($\mathcal{J}$-self-adjoint), respectively, in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

**Proposition 2.6.1** *A block operator matrix $\mathcal{A}$ is symmetric if and only if*

$$A \subset A^*, \quad D \subset D^*, \quad C|_{\mathcal{D}(A)\cap\mathcal{D}(C)} \subset B^*, \qquad B|_{\mathcal{D}(B)\cap\mathcal{D}(D)} \subset C^*, \tag{2.6.1}$$

*and $\mathcal{J}$-symmetric if and only if*

$$A \subset A^*, \quad D \subset D^*, \quad C|_{\mathcal{D}(A)\cap\mathcal{D}(C)} \subset -B^*, \quad B|_{\mathcal{D}(B)\cap\mathcal{D}(D)} \subset -C^*.$$

***Proof.*** The operator matrix $\mathcal{A}$ is symmetric if and only if, for all elements $\mathbf{x} = (x_1\ x_2)^{\mathrm{t}}$, $\mathbf{y} = (y_1\ y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{A}) = \big(\mathcal{D}(A)\cap\mathcal{D}(C)\big) \oplus \big(\mathcal{D}(B)\cap\mathcal{D}(D)\big)$, we have $(\mathcal{A}\mathbf{x},\mathbf{y}) = (\mathbf{x},\mathcal{A}\mathbf{y})$ or, equivalently,

$$\begin{aligned}&(Ax_1,y_1)+(Bx_2,y_1)+(Cx_1,y_2)+(Dy_1,y_2)\\ &=(x_1,Ay_1)+(x_2,Cy_1)+(x_1,By_2)+(y_1,Dy_2).\end{aligned} \tag{2.6.2}$$

Setting $x_2 = y_2 = 0$ and $x_1 = y_1 = 0$ in (2.6.2) shows that $A$ and $D$ are symmetric. For $y_2 = 0$, (2.6.2) yields that

$$(Ax_1,y_1) + (Bx_2,y_1) = (x_1,Ay_1) + (x_2,Cy_1).$$

The symmetry of $A$ implies $y_1 \in \mathcal{D}(B^*)$, $B^*y_1 = Cy_1$, *i.e.* $C|_{\mathcal{D}(A)\cap\mathcal{D}(C)} \subset B^*$; the proof of the remaining inclusion is similar if we choose $y_1 = 0$ in (2.6.2).

Conversely, if (2.6.1) holds, it is easy to check that (2.6.2) is satisfied, and hence $\mathcal{A}$ is symmetric.

The second assertion follows from the first claim applied to $\mathcal{J}\mathcal{A}$. □

**Corollary 2.6.2** *If $\mathcal{D}(A)\cap\mathcal{D}(C)$ is a core of $C$ or if $\mathcal{D}(B)\cap\mathcal{D}(D)$ is a core of $B$, then $\mathcal{A}$ is symmetric if and only if*

$$A \subset A^*, \quad D \subset D^*, \quad C \subset B^*, \tag{2.6.3}$$

*and $\mathcal{J}$-symmetric if and only if*

$$A \subset A^*, \quad D \subset D^*, \quad C \subset -B^*.$$

***Proof.*** If (2.6.3) holds, then $B \subset \overline{B} = (B^*)^* \subset C^*$ and so (2.6.1) follows. Vice versa, if $\mathcal{D}(A)\cap\mathcal{D}(C)$ is a core of $C$, then (2.6.1) implies that $C \subset \overline{C} = \overline{C|_{\mathcal{D}(A)\cap\mathcal{D}(C)}} \subset B^*$ since $B^*$ is closed; if $\mathcal{D}(B)\cap\mathcal{D}(D)$ is a core of $B$, then (2.6.1) implies that $B \subset \overline{B} = \overline{B|_{\mathcal{D}(B)\cap\mathcal{D}(D)}} \subset C^*$ since $C^*$ is closed and thus, taking adjoints, $C \subset \overline{C} = (C^*)^* \subset B^*$. The second claim follows from the first applied to $\mathcal{J}\mathcal{A}$. □

In order to obtain criteria for self-adjointness ($\mathcal{J}$-self-adjointness), we consider block diagonal and block off-diagonal operator matrices.

**Proposition 2.6.3** *For the block operator matrices*

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

*we have*

$$\mathcal{T}^* := \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}, \quad \mathcal{S}^* := \begin{pmatrix} 0 & C^* \\ B^* & 0 \end{pmatrix},$$

*and hence*

i) *$\mathcal{T}$ is self-adjoint if and only if $A = A^*$ and $D = D^*$,*
ii) *$\mathcal{S}$ is self-adjoint if and only if $B$ is closed and $C = B^*$;*

*in the latter case, we have $\rho(\mathcal{S}) \setminus \{0\} = \{\lambda \in \mathbb{C} : \lambda^2 \in \rho(B^*B)\}$, $0 \in \rho(\mathcal{S})$ if and only if $B$ is bijective, and the resolvent of $\mathcal{S}$ is given by*

$$(\mathcal{S} - \lambda)^{-1} = \begin{pmatrix} \lambda(BB^* - \lambda^2)^{-1} & B(B^*B - \lambda^2)^{-1} \\ B^*(BB^* - \lambda^2)^{-1} & \lambda(B^*B - \lambda^2)^{-1} \end{pmatrix}, \quad \lambda \in \rho(\mathcal{S}). \quad (2.6.4)$$

***Proof.*** The claims for $\mathcal{T}$ are obvious.

An element $(y_1\ y_2)^{\mathrm{t}} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ belongs to $\mathcal{D}(\mathcal{S}^*)$ if and only if there exist $(y_1^*\ y_2^*)^{\mathrm{t}} \in \mathcal{H}_1 \oplus \mathcal{H}_2$ such that for all $(x_1\ x_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}) = \mathcal{D}(C) \oplus \mathcal{D}(B)$

$$\left( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \right)$$
$$\iff \quad (Cx_1, y_2) + (Bx_2, y_1) = (x_1, y_1^*) + (x_2, y_2^*).$$

If $y_1 \in \mathcal{D}(B^*)$, $y_2 \in \mathcal{D}(C^*)$, the latter holds with $y_1^* = C^*y_2$, $y_2^* = B^*y_1$; this proves that $\mathcal{D}(B^*) \oplus \mathcal{D}(C^*) \subset \mathcal{D}(\mathcal{S}^*)$ and that $\mathcal{S}^*|_{\mathcal{D}(B^*)\oplus\mathcal{D}(C^*)}$ has the asserted form. Vice versa, if $(y_1\ y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}^*)$, we choose $(x_1\ 0)^{\mathrm{t}}, (0\ x_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}) = \mathcal{D}(C) \oplus \mathcal{D}(B)$ above and obtain $y_1 \in \mathcal{D}(B^*)$, $y_2 \in \mathcal{D}(C^*)$.

From the formula for $\mathcal{S}^*$, claim ii) is immediate. Moreover, if $\lambda \in \mathbb{C}\setminus\{0\}$ is such that $\lambda^2 \in \rho(B^*B)$ or, equivalently, $\lambda^2 \in \rho(BB^*)$, it is not difficult to check that the resolvent of $\mathcal{S}$ is given by the above formula; note that $(BB^*-\lambda^2)^{-1}B^* = B^*(B^*B-\lambda^2)^{-1}$ and $(B^*B-\lambda^2)^{-1}B = B(BB^*-\lambda^2)^{-1}$. Vice versa, if $\lambda \in \rho(\mathcal{S}) \setminus \{0\}$, the unique solvability of $(\mathcal{S} - \lambda)\mathbf{x} = \mathbf{f}$ for $\mathbf{f} = (f\ 0)^{\mathrm{t}}$ shows that $\lambda^2 \in \rho(B^*B)$; for $\lambda = 0$, the assertion is clear. □

**Remark 2.6.4** The self-adjointness of the block off-diagonal matrix $\mathcal{S}$ can be used to prove von Neumann's theorem (see [Kat95, Theorem V.3.24]: If $B$ is closed, then $B^*B$ is self-adjoint. In fact, $BB^*$ and $B^*B$ are the diagonal entries of the self-adjoint block diagonal operator $\mathcal{S}^2$. This proof is known as Nelson's trick (see [Tha92, Sections 5.2.2, 5.2.3]).

The next theorem shows that, under certain assumptions, the adjoint of an unbounded block operator matrix is again a block operator matrix.

**Theorem 2.6.5** *Suppose that block operator matrix $\mathcal{A}$ satisfies one of the following two conditions:*

i) *$A$, $D$ are Fredholm operators, $C$ is $A$-compact, and $B$ is $D$-compact;*
ii) *$B$, $C$ are Fredholm operators, $A$ is $C$-compact, and $D$ is $B$-compact.*

*Then the adjoint of $\mathcal{A}$ is given by*

$$\mathcal{A}^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

***Proof.*** We prove the claim in case i); the proof in case ii) is analogous. Let the block operator matrices $\mathcal{T}$ and $\mathcal{S}$ be defined as in Proposition 2.6.3. Then $\mathcal{T}$ is Fredholm and, by Theorem 2.4.1 (with $C_0 = B_0 = A_1 = D_1 = 0$), $\mathcal{S}$ is $\mathcal{T}$-compact. Now the claim follows from Proposition 2.1.11 (see also the original result in [Bea64]) together with Proposition 2.6.3. □

**Theorem 2.6.6** *Suppose that $\mathcal{A}$ is diagonally dominant of order $\delta < 1$ and that $A = A^*$ and $D = D^*$.*

i) *If $C \subset B^*$, then $\mathcal{A}$ is self-adjoint in $\mathcal{H}$.*
ii) *If $C \subset -B^*$, then $\mathcal{A}$ is $\mathcal{J}$-self-adjoint in $\mathcal{H}$.*

*Suppose that $\mathcal{A}$ is off-diagonally dominant of order $\delta < 1$ and $B$ is closed.*

iii) *If $C = B^*$, then $\mathcal{A}$ is self-adjoint in $\mathcal{H}$.*
iv) *If $C = -B^*$, then $\mathcal{A}$ is $\mathcal{J}$-self-adjoint in $\mathcal{H}$.*

***Proof.*** i), iii) By the assumptions, the block diagonal and off-diagonal operators $\mathcal{T}$ and $\mathcal{S}$, respectively, defined in Proposition 2.6.3 are self-adjoint. The claims now follow from Proposition 2.2.5 and the theorem of Kato-Rellich (see [Kat95, Theorem V.4.3]), which shows that self-adjointness is preserved under relatively bounded perturbations with relative bound $< 1$.

ii), iv) The claims follow from i), iii) applied to $\mathcal{J}\mathcal{A}$. □

The main result of this section shows that if the quadratic numerical range of a block operator matrix $\mathcal{A}$ is real, then the diagonal entries $A$ and $D$ of $\mathcal{A}$ are symmetric and $\mathcal{A}$ is either block triangular, similar to a symmetric block operator matrix, or similar to a $\mathcal{J}$-symmetric block operator matrix.

Note that the quadratic numerical range of a $\mathcal{J}$-self-adjoint block operator matrix may be complex even for bounded $B$ having sufficiently large norm.

**Theorem 2.6.7** *Suppose that* $\dim \mathcal{H}_1 \geq 2$ *or* $\dim \mathcal{H}_2 \geq 2$, $\mathcal{D}(B) \cap \mathcal{D}(D)$ *is a core of* $B$, *and* $\mathcal{D}(A) \cap \mathcal{D}(C)$ *is a core of* $C$. *If* $W^2(\mathcal{A}) \subset \mathbb{R}$, *then* $A$ *and* $D$ *are symmetric and* $\mathcal{A}$ *is either block triangular* (*i.e.* $B = 0$ *or* $C = 0$) *or there exists a* $\gamma \in \mathbb{R}$, $\gamma \neq 0$, *such that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad B \neq 0, \quad C \subset \gamma B^*;$$

*in the latter case,* $\mathcal{A}$ *is similar to the block operator matrix*

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & \widetilde{B} \\ \widetilde{C} & D \end{pmatrix}, \quad \widetilde{B} := \sqrt{|\gamma|}\, B, \quad \widetilde{C} \subset (\operatorname{sign} \gamma)\, \widetilde{B}^*;$$

$\widetilde{\mathcal{A}}$ *is symmetric in* $\mathcal{H}$ *if* $\operatorname{sign} \gamma = 1$ *and* $\mathcal{J}$*-symmetric if* $\operatorname{sign} \gamma = -1$.

***Proof.*** We follow the lines of the proof of Theorem 1.1.15 in the bounded case with the following modifications. By Theorem 2.5.4, we have $W(A) \subset W^2(\mathcal{A}) \subset \mathbb{R}$ and hence $A$ is symmetric and, as a consequence, so is $D$. In the same way as in the proof of Theorem 2.5.4, we conclude that $(Bg, f)(Cf, g) \in \mathbb{R}$ for all $f \in \mathcal{D}(A) \cap \mathcal{D}(C)$, $g \in \mathcal{D}(B) \cap \mathcal{D}(D)$. Since $\mathcal{D}(A) \cap \mathcal{D}(C)$ is a core of $C$ and $C$ is closed, we have $C = \overline{C} = \overline{C|_{\mathcal{D}(A)\cap\mathcal{D}(C)}}$, and analogously for $\mathcal{D}(B) \cap \mathcal{D}(D)$ and $B$. Hence, for $x \in \mathcal{D}(C)$, $y \in \mathcal{D}(B)$, there exist sequences $(x_n)_1^\infty \subset \mathcal{D}(A) \cap \mathcal{D}(C)$, $(y_n)_1^\infty \subset \mathcal{D}(B) \cap \mathcal{D}(D)$ such that $x_n \to x$, $y_n \to y$ and $Cx_n \to \overline{C}x = Cx$, $By_n \to \overline{B}y = By$ for $n \to \infty$. This shows that, in fact, $(Bg, f)(Cf, g) \in \mathbb{R}$ for all $f \in \mathcal{D}(C)$, $g \in \mathcal{D}(B)$.

Now Lemma 1.1.16, which was already formulated for unbounded operators, shows that the first claim holds. The similarity of $\mathcal{A}$ to $\widetilde{\mathcal{A}}$ is proved in the same way as in the proof of Theorem 1.1.15. The last claim about the symmetry ($\mathcal{J}$-symmetry) of $\mathcal{A}$ follows from Corollary 2.6.2. □

For unbounded $\mathcal{J}$-self-adjoint block operator matrices, the quadratic numerical range yields the following spectral enclosures (see [Tre08, Theorem 5.4] and compare Proposition 1.3.9 for the bounded case).

Here and in the following, we use the definitions of the functionals $\lambda_\pm$ and their ranges $\Lambda_\pm(\mathcal{A})$ given in Corollary 1.1.4 with the obvious domain restrictions for the unbounded case (see Remark 2.5.2).

**Proposition 2.6.8** *Let* $A = A^*$, $D = D^*$, $C = -B^*$, *and let* $\mathcal{A}$ *be either diagonally dominant or off-diagonally dominant of order* 0 (*so that* $\mathcal{A}$ *is* $\mathcal{J}$*-self-adjoint*), *and, in the latter case, let* $B$ *be boundedly invertible.*

i) *If* $W^2(\mathcal{A}) \subset \mathbb{R}$, *then* $\overline{W^2(\mathcal{A})}$ *consists of one or two intervals and*

$$\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})} \subset \overline{\Lambda_-(\mathcal{A})} \cup \overline{\Lambda_+(\mathcal{A})}.$$

ii) *If $A$ and $D$ are bounded from below,*

$$a_- := \inf W(A), \quad d_- := \inf W(D),$$

*then*

$$\sigma(\mathcal{A}) \cap \mathbb{R} \subset \big[\min\{a_-, d_-\}, \infty\big), \quad \sigma(\mathcal{A}) \backslash \mathbb{R} \subset \Big\{z \in \mathbb{C} : \frac{a_- + d_-}{2} \leq \operatorname{Re} z\Big\};$$

*analogous statements hold for $A$ and $D$ bounded from above.*

iii) *If $B$ is bounded, then $\sigma(\mathcal{A}) \setminus \mathbb{R} \subset \big\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \|B\|\big\}$; if, in addition, $\delta := \operatorname{dist}\big(W(A), W(D)\big) > 0$, then*

$$\|B\| \leq \delta/2 \implies \sigma(\mathcal{A}) \subset \mathbb{R},$$
$$\|B\| > \delta/2 \implies \sigma(\mathcal{A}) \setminus \mathbb{R} \subset \big\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \sqrt{\|B\|^2 - \delta^2/4}\big\}.$$

***Proof.*** In a similar way as in the proof of Theorem 2.5.18, we consider the block operator matrices

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix};$$

$\mathcal{T}$ is self-adjoint and, if $B$ is boundedly invertible and thus closed, so is $\mathrm{i}\mathcal{S}$.

i) If $W^2(\mathcal{A}) \subset \mathbb{R}$, its at most two components must be intervals. By Corollary 2.5.17, it suffices to prove that there exist points $\mu_+, \mu_- \in \rho(\mathcal{A})$ in the upper and lower half-plane, respectively. To this end, we note that

$$\big\|(\mathcal{T} - \mathrm{i}\mu)^{-1}\big\| \leq \frac{1}{|\mu|}, \qquad \mu \in \mathbb{R} \setminus \{0\},$$
$$\big\|\big(\mathcal{S} - (\nu \pm \mathrm{i}\mu_0)\big)^{-1}\big\| \leq \frac{1}{|\nu|}, \quad \nu \in \mathbb{R} \setminus \{0\},$$

with an arbitrary fixed $\mu_0 \in (0, \infty)$. As in the proof of Theorem 2.5.18, we find that the assumptions of Corollary 2.1.5 are satisfied for $\mathcal{T}$ and $\mathcal{S} \mp \mathrm{i}\mu_0$, respectively. As a consequence, there exists an $R > 0$ such that $\{\pm \mathrm{i}\mu : |\mu| \geq R\} \subset \rho(\mathcal{T} + \mathcal{S}) = \rho(\mathcal{A})$ in the diagonally dominant case and $\{\nu \pm \mathrm{i}\mu_0 : |\nu| \geq R\} \subset \rho(\mathcal{S} + \mathcal{T}) = \rho(\mathcal{A})$ in the off-diagonally dominant case.

ii) As in the bounded case, we can show that the asserted inclusions hold with $W^2(\mathcal{A})$ instead of $\sigma(\mathcal{A})$ (see Propositions 1.2.6 and 1.3.9). By Corollary 2.5.17, it suffices to prove that there is a point $\mu \in \rho(\mathcal{A})$ in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < (a_- + d_-)/2\}$. Here we note that

$$\big\|\big(\mathcal{T} - (\nu_0 + \mathrm{i}\mu)\big)^{-1}\big\| \leq \frac{1}{|\mu|}, \quad \mu \in \mathbb{R} \setminus \{0\},$$
$$\big\|(\mathcal{S} - \mu)^{-1}\big\| \leq \frac{1}{|\mu|}, \qquad \mu \in \mathbb{R} \setminus \{0\},$$

with an arbitrary fixed $\nu_0 \in \big(-\infty, a_- + d_-)/2\big)$. Hence, by Corollary 2.1.5 applied to $\mathcal{T} - \nu_0$ and $\mathcal{S}$, respectively, there exists an $R > 0$ such that

$\{\nu_0 + \mathrm{i}\mu : |\mu| \geq R\} \subset \rho(\mathcal{T}+\mathcal{S}) = \rho(\mathcal{A})$ in the diagonally dominant case and $\{\pm\mu : |\mu| \geq R\} \subset \rho(\mathcal{S}+\mathcal{T}) = \rho(\mathcal{A})$ in the off-diagonally dominant case.

iii) If $B$ is bounded, then $\mathcal{A}$ is a bounded perturbation of the self-adjoint operator $\mathcal{T}$ and the first claim follows from standard perturbation theorems (see *e.g.* [Kat95, Problem V.4.8]); in particular, there are $\mu_+, \mu_- \in \rho(\mathcal{A})$ in the upper and lower half-plane, respectively. Now the claims for $\delta > 0$ follow in the same way as in the bounded case (see Propositions 1.2.6, 1.3.9). □

For diagonally dominant block operator matrices, the fact that $W^2(\mathcal{A})$ is real is related to certain definiteness properties of the Schur complements. For this we first prove the following stability result.

**Lemma 2.6.9** *Let $\mathcal{A}$ be diagonally dominant, $A = A^*$, $D = D^*$, and either $C \subset B^*$ or $C \subset -B^*$. If $\lambda \in \rho(D) \cap \mathbb{R}$, then $S_1(\lambda)$ is symmetric. If there exists a $\lambda_0 \in \rho(A) \cap \rho(D) \cap \mathbb{R}$ so that $S_1(\lambda_0)$ is self-adjoint and uniformly positive, then there is an $\varepsilon > 0$ so that $S_1(\lambda)$ is self-adjoint and uniformly positive for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Analogous statements hold for $S_2$.*

***Proof.*** The symmetry of $S_1(\lambda) = A - \lambda \mp B(D-\lambda)^{-1}B^*$ is obvious. By Proposition 2.3.4, the set $\rho(S_1) \cap \rho(A)$ is open. Hence there exists a $\delta > 0$ such that $(\lambda_0 - \delta, \lambda_0 + \delta) \subset \rho(S_1) \cap \rho(A)$. For $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ and $x \in \mathcal{D}\big(S_1(\lambda)\big) = \mathcal{D}\big(S_1(\lambda_0)\big) = \mathcal{D}(A)$, we have the estimate

$$\begin{aligned}
&\big\|\big(S_1(\lambda) - S_1(\lambda_0)\big)x\big\| \\
&= |\lambda - \lambda_0|\, \big\|\big(I \mp B(D-\lambda)^{-1}(D-\lambda_0)^{-1}B^*\big)x\big\| \\
&\leq |\lambda - \lambda_0| \big(\|x\| + \|B(D-\lambda)^{-1}\|\, \|(D-\lambda_0)^{-1}\|\, \|B^*(A-\lambda_0)^{-1}\|\, \|(A-\lambda_0)x\|\big).
\end{aligned}$$

Since $\lambda_0 \in \rho(S_1) \cap \rho(A)$, the operator $I \mp B(D-\lambda_0)^{-1}B^*(A-\lambda_0)^{-1}$ is boundedly invertible by Proposition 2.3.4 and hence

$$A - \lambda_0 = \big(I \mp B(D-\lambda_0)^{-1}B^*(A-\lambda_0)^{-1}\big)^{-1} S_1(\lambda_0).$$

Together with the fact that $\mu \mapsto B(D-\mu)^{-1}$ is holomorphic on $\rho(D)$, we conclude that there exists a $c_\delta > 0$ such that, for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$,

$$\big\|\big(S_1(\lambda) - S_1(\lambda_0)\big)x\big\| \leq |\lambda - \lambda_0|\, \|x\| + |\lambda - \lambda_0|\, c_\delta\, \|S_1(\lambda_0)x\|. \qquad (2.6.5)$$

Now a stability theorem for semi-bounded self-adjoint operators (see [Kat95, Theorem V.4.11]) shows that if $|\lambda - \lambda_0| < 1/c_\delta$, then $S_1(\lambda) - S_1(\lambda_0)$ is $S_1(\lambda_0)$-bounded with $S_1(\lambda_0)$-bound $< 1$ and hence $S_1(\lambda)$ is self-adjoint; moreover, if $S_1(\lambda_0) \geq s_0 > 0$, then (2.6.5) implies that $S_1(\lambda) \geq s_1$ with

$$s_1 = s_0 - \max\left\{ \frac{|\lambda - \lambda_0|}{1 - |\lambda - \lambda_0| c_\delta},\ |\lambda - \lambda_0|\, \big(1 + c_\delta s_0\big) \right\}$$

(see [Kat95, (V.4.13)]). Hence there exists an $\varepsilon > 0$, $\varepsilon < \min\{\delta, 1/c_\delta\}$, such that $s_1 > 0$ for $|\lambda - \lambda_0| < \varepsilon$. □

**Proposition 2.6.10** *Let $\mathcal{A}$ be diagonally dominant of order* 0*,* $A = A^*$*,* $D = D^*$*,* $C \subset -B^*$ *(so that $\mathcal{A}$ is $\mathcal{J}$-self-adjoint), and let $\gamma \in \mathbb{R}$ be such that*

$$\max \sigma(D) < \gamma < \min \sigma(A). \tag{2.6.6}$$

*Then the following are equivalent:*

(i) $W^2(\mathcal{A}) \subset \mathbb{R}$ *and* $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$,
(ii) $S_1(\gamma)$ *is self-adjoint and uniformly positive,*
(iii) $S_2(\gamma)$ *is self-adjoint and uniformly negative.*

***Proof.*** In the following, we prove the equivalence of (i) and (ii). The proof of the equivalence of (i) and (iii) is similar.

Suppose that (i) holds and let $x \in \mathcal{D}\big(S_1(\gamma)\big) = \mathcal{D}(A)$, $x \neq 0$. If $B^*x = 0$, then $\big(S_1(\gamma)x, x\big) = (Ax, x) - \gamma \|x\|^2 > 0$ by (2.6.6). If $B^*x \neq 0$, set $y := -(D-\gamma)^{-1}B^*x$. By definition, $\lambda_\pm(x, y)$ are the zeroes of the function $\Delta(x, y; \cdot)$ in Lemma 2.5.7. Hence, by (2.5.3) therein and by (i), we obtain

$$\begin{aligned} 0 &> \left(\gamma - \lambda_+\begin{pmatrix} x \\ y \end{pmatrix}\right)\left(\gamma - \lambda_-\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= \frac{1}{\|x\|^2 \|(D-\gamma)^{-1}B^*x\|^2}\, \Delta\big(x, (D-\gamma)^{-1}B^*x; \gamma\big) \\ &= \frac{1}{\|x\|^2 \|(D-\gamma)^{-1}B^*x\|^2}\, \big((D-\gamma)^{-1}B^*x, B^*x\big)\big(S_1(\gamma)x, x\big). \end{aligned}$$

Here the first factor is positive, the second factor is negative by (2.6.6), and hence $\big(S_1(\gamma)x, x\big) > 0$. By Theorem 2.6.8 i), the assumptions (2.6.6) and (i) imply that $\gamma \in \rho(\mathcal{A}) \setminus \sigma(D)$ and hence $\gamma \in \rho(S_1)$ or, equivalently, $0 \in \rho\big(S_1(\gamma)\big)$ by Theorem 2.3.3 ii). This proves (ii).

Conversely, assume that (ii) holds. By Lemma 2.6.9, there exists an $\varepsilon > 0$ with $\max \sigma(D) < \gamma - \varepsilon < \gamma + \varepsilon < \min \sigma(A)$ and such that $S_1(\lambda)$ is self-adjoint and uniformly positive for all $\lambda \in (\gamma - \varepsilon, \gamma + \varepsilon)$. Now let $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(D)$, $\|x\| = \|y\| = 1$, and $\lambda \in (\gamma - \varepsilon, \gamma + \varepsilon)$. Using the Cauchy–Schwarz inequality with respect to the scalar product $\big((\lambda - D)^{-1}\cdot, \cdot\big)$, we find that

$$\begin{aligned} &\left(\lambda - \lambda_+\begin{pmatrix} x \\ y \end{pmatrix}\right)\left(\lambda - \lambda_-\begin{pmatrix} x \\ y \end{pmatrix}\right) = \Delta(x, y; \lambda) \\ &= \big((\lambda - A)x, x\big)\big((\lambda - D)y, y\big) + \big|(B^*x, y)\big|^2 \\ &\le \big((\lambda - A)x, x\big)\big((\lambda - D)y, y\big) + \big((\lambda - D)^{-1}B^*x, B^*x\big)\big((\lambda - D)y, y\big) \\ &= -\big((\lambda - D)y, y\big)\big(S_1(\lambda)x, x\big) < 0 \end{aligned}$$

and, consequently, $\lambda_-(x, y) < \lambda < \lambda_+(x, y)$. This proves the inequalities $\lambda_-(x, y) \le \gamma - \varepsilon < \gamma + \varepsilon \le \lambda_+(x, y)$ and hence (i). □

The inclusion $\sigma_{\rm p}(\mathcal{A}) \subset W^2(\mathcal{A})$ opens up a way to classify eigenvectors and eigenvalues of diagonally dominant self-adjoint and $\mathcal{J}$-self-adjoint block operator matrices. Using the functionals $\lambda_\pm$, we distinguish eigenvectors of positive, negative, and neutral type.

**Definition 2.6.11** Suppose that $\mathcal{A}$ is self-adjoint or $\mathcal{J}$-self-adjoint. Let $\lambda_0 \in \sigma_{\rm p}(\mathcal{A})$ be an eigenvalue of $\mathcal{A}$ and let $\mathbf{x}_0 = (x_0\ \ y_0)^{\rm t} \in \mathcal{D}(\mathcal{A})$ be a corresponding eigenvector such that $x_0,\, y_0 \neq 0$. Then $\mathbf{x}_0$ is said to be of

$$\begin{aligned}&\textit{positive type} \text{ if } \lambda_0 = \lambda_+(\mathbf{x}_0) \text{ and } \lambda_+(\mathbf{x}_0) \neq \lambda_-(\mathbf{x}_0),\\ &\textit{negative type} \text{ if } \lambda_0 = \lambda_-(\mathbf{x}_0) \text{ and } \lambda_-(\mathbf{x}_0) \neq \lambda_+(\mathbf{x}_0),\\ &\textit{neutral type} \text{ if } \lambda_0 = \lambda_+(\mathbf{x}_0) = \lambda_-(\mathbf{x}_0);\end{aligned}$$

the eigenvector $\mathbf{x}_0$ is said to be of *definite type* if it is either of positive or of negative type. If $\lambda_0$ is (algebraically) simple, we call $\lambda_0$ of positive, negative, definite, or neutral type if so is the corresponding eigenvector.

In the following cases where $\overline{W^2(\mathcal{A})}$ is real and consists of two disjoint intervals it is easy to determine the types of the eigenvalues of $\mathcal{A}$.

**Proposition 2.6.12** *Let $\mathcal{A}$ be self-adjoint or $\mathcal{J}$-self-adjoint and let $\lambda_0 \in \sigma_{\rm p}(\mathcal{A})$ be simple with eigenvector $\mathbf{x}_0 = (x_0\ \ y_0)^{\rm t} \in \mathcal{D}(\mathcal{A})$ so that $x_0,\, y_0 \neq 0$.*

i) *If $\mathcal{A}$ is self-adjoint and $\gamma \in \mathbb{R}$ is so that $\sup W(D) < \gamma < \inf W(A)$, then*

$$\begin{aligned}\lambda_0 \in (\gamma, \infty) \quad &\Longrightarrow \quad \lambda_0 \ \textit{is of positive type},\\ \lambda_0 \in (-\infty, \gamma) \quad &\Longrightarrow \quad \lambda_0 \ \textit{is of negative type}.\end{aligned}$$

ii) *If $\mathcal{A}$ is $\mathcal{J}$-self-adjoint, $W^2(\mathcal{A}) \subset \mathbb{R}$, and there exists a $\gamma \in \mathbb{R}$ such that $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$, then*

$$\begin{aligned}\lambda_0 \in (\gamma, \infty) \quad &\Longrightarrow \quad \lambda_0 \ \textit{is of negative type},\\ \lambda_0 \in (-\infty, \gamma) \quad &\Longrightarrow \quad \lambda_0 \ \textit{is of positive type}.\end{aligned}$$

***Proof.*** In both cases, the claims are immediate from the two inclusions $\sigma_{\rm p}(\mathcal{A}) \cap (\gamma, \infty) \subset \Lambda_+(\mathcal{A})$ and $\sigma_{\rm p}(\mathcal{A}) \cap (-\infty, \gamma) \subset \Lambda_-(\mathcal{A})$; the latter are obvious for ii) and were proved in Remark 2.5.19 for i). □

Next we prove that, for small perturbations $tB$ of $B$, simple isolated eigenvalues of definite type move in opposite directions for self-adjoint and for $\mathcal{J}$-self-adjoint block operator matrices: for self-adjoint $\mathcal{A}$, positive type eigenvalues move to the right and negative type eigenvalues move to the left, whereas for $\mathcal{J}$-self-adjoint $\mathcal{A}$, they move exactly in the opposite directions.

**Theorem 2.6.13** *Suppose that $\mathcal{A}$ is self-adjoint or $\mathcal{J}$-self-adjoint, and let $\lambda_0 \in \mathbb{R}$ be an isolated simple eigenvalue of $\mathcal{A}$ with eigenvector $\mathbf{x}_0 = (x_0\ \ y_0)^{\mathrm{t}}$. Then there exists an $\varepsilon > 0$ such that for all $t \in (1-\varepsilon, 1+\varepsilon)$ the operator*

$$\mathcal{A}_t := \begin{pmatrix} A & tB \\ tC & D \end{pmatrix} \tag{2.6.7}$$

*has exactly one eigenvalue $\lambda(t)$ in a neighbourhood of $\lambda_0$. If $x_0, y_0 \neq 0$, then*

$$\mathbf{x}_0 \ \textit{is of positive type} \quad \Longleftrightarrow \quad \begin{cases} \lambda'(1) > 0 & \textit{if } C \subset B^*, \\ \lambda'(1) < 0 & \textit{if } C \subset -B^*, \end{cases}$$

$$\mathbf{x}_0 \ \textit{is of negative type} \quad \Longleftrightarrow \quad \begin{cases} \lambda'(1) < 0 & \textit{if } C \subset B^*, \\ \lambda'(1) > 0 & \textit{if } C \subset -B^*, \end{cases}$$

$$\mathbf{x}_0 \ \textit{is of neutral type} \quad \Longleftrightarrow \quad \lambda'(1) = 0.$$

*If $x_0 = 0$ or $y_0 = 0$, then $\lambda'(1) = 0$.*

***Proof.*** Obviously, the operator $\mathcal{A}_t$ can be written as

$$\mathcal{A}_t = \mathcal{A} + (t-1)\mathcal{S}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad t \in \mathbb{C}.$$

Since $\mathcal{D}(\mathcal{A}_t) = \mathcal{D}(\mathcal{A})$ is independent of $t$ for all $t$ in a neighbourhood of 1 and $\mathcal{A}_t$ depends holomorphically on $t$, the operators $\mathcal{A}_t$ form a holomorphic family of type (A) (see [Kat95, Section VII.2.1]).

First we consider the self-adjoint case where that $C \subset B^*$. By [Kat95, Section VII.3.4, (3.18)], it follows that

$$\lambda'(1) = \frac{(\mathcal{S}\mathbf{x}_0, \mathbf{x}_0)}{\|\mathbf{x}_0\|^2} = \frac{2\,\mathrm{Re}\,(By_0, x_0)}{\|\mathbf{x}_0\|^2}. \tag{2.6.8}$$

The case $x_0 = 0$ or $y_0 = 0$ is clear from (2.6.8). Now let $x_0, y_0 \neq 0$. The eigenvalue equation $(\mathcal{A} - \lambda_0)\mathbf{x}_0 = 0$ implies the relations

$$\frac{(Ax_0, x_0)}{\|x_0\|^2} - \lambda_0 + \frac{(By_0, x_0)}{\|x_0\|^2} = 0, \tag{2.6.9}$$

$$\frac{(Dy_0, y_0)}{\|y_0\|^2} - \lambda_0 + \frac{(B^*x_0, y_0)}{\|y_0\|^2} = 0; \tag{2.6.10}$$

in particular, since $A$ and $D$ are self-adjoint, $(By_0, x_0)$ is real and hence $(By_0, x_0) = (B^*x_0, y_0)$. Adding the two equations (2.6.9), (2.6.10), we find

$$\begin{aligned} \left( \frac{1}{\|x_0\|^2} + \frac{1}{\|y_0\|^2} \right) (By_0, x_0) &= 2\lambda_0 - \left( \frac{(Ax_0, x_0)}{\|x_0\|^2} + \frac{(Dy_0, y_0)}{\|y_0\|^2} \right) \\ &= 2\lambda_0 - \big(\lambda_+(\mathbf{x}_0) + \lambda_-(\mathbf{x}_0)\big). \end{aligned}$$

Together with (2.6.8), this yields

$$\lambda'(1) = \frac{4\,\|x_0\|^2\|y_0\|^2}{(\|x_0\|^2+\|y_0\|^2)^2}\left(\lambda_0 - \frac{\lambda_+(\mathbf{x}_0)+\lambda_-(\mathbf{x}_0)}{2}\right). \tag{2.6.11}$$

Clearly, the relations (2.6.9), (2.6.10) imply that $\det\big(\mathcal{A}_{x_0,y_0}-\lambda_0\big)=0$ and so $\lambda_0=\lambda_+(\mathbf{x}_0)$ or $\lambda_0=\lambda_-(\mathbf{x}_0)$. This, together with (2.6.11) and the inequality $\lambda_-(\mathbf{x}_0)\leq\lambda_+(\mathbf{x}_0)$, yields the desired equivalences if $\mathcal{A}$ is self-adjoint.

Now suppose that $\mathcal{A}$ is $\mathcal{J}$-self-adjoint and hence $C \subset -B^*$. Since $\lambda_0$ is a real (algebraically) simple and isolated eigenvalue of the $\mathcal{J}$-self-adjoint operator $\mathcal{A}$, the corresponding eigenvector $\mathbf{x_0}$ is non-degenerate (see *e.g.* [Bog74, Corollary VI.6.6]), that is, $[\mathbf{x_0},\mathbf{x_0}] = \|x_0\|^2 - \|y_0\|^2 \neq 0$. The analogue of [Kat95, Section VII.3.4, (3.18)] for the $\mathcal{J}$-self-adjoint perturbation $\mathcal{S}$ is given by

$$\lambda'(1) = \frac{\big[\mathcal{S}\mathbf{x}_0,\mathbf{x}_0\big]}{[\mathbf{x}_0,\mathbf{x}_0]} = \frac{2\,\mathrm{Re}\,(By_0,x_0)}{\|x_0\|^2-\|y_0\|^2}; \tag{2.6.12}$$

here the first formula is obtained in a similar way as [Kat95, Section II.6.5, (6.10)] by taking the indefinite inner product $[\cdot,\cdot]$ with the eigenvector $\mathbf{x_0}$ and using the symmetry of $\mathcal{S}$ with respect to $[\cdot,\cdot]$. The case $x_0 = 0$ or $y_0 = 0$ is clear from (2.6.12). Now let $x_0, y_0 \neq 0$. The eigenvalue equation $(\mathcal{A}-\lambda_0)\mathbf{x}_0 = 0$ implies the relations

$$\frac{(Ax_0,x_0)}{\|x_0\|^2} - \lambda_0 + \frac{(By_0,x_0)}{\|x_0\|^2} = 0, \tag{2.6.13}$$

$$\frac{(Dy_0,y_0)}{\|y_0\|^2} - \lambda_0 - \frac{(B^*x_0,y_0)}{\|y_0\|^2} = 0. \tag{2.6.14}$$

In the same way as in the self-adjoint case, we find that

$$\lambda'(1) = -\frac{4\,\|x_0\|^2\|y_0\|^2}{(\|x_0\|^2-\|y_0\|^2)^2}\left(\lambda_0 - \frac{\lambda_+(\mathbf{x}_0)+\lambda_-(\mathbf{x}_0)}{2}\right),$$

so, compared to the self-adjoint case, the inequality signs get reversed because of the minus sign in front of the right hand side. □

**Remark 2.6.14** If, in Theorem 2.6.13, $\mathcal{A}$ is $\mathcal{J}$-self-adjoint, $\overline{W(D)} < \overline{W(A)}$, and $x_0$, $y_0 \neq 0$, then the first two equivalences can be continued as

$$\lambda'(1) < 0 \iff [\mathbf{x}_0,\mathbf{x}_0] > 0,$$
$$\lambda'(1) > 0 \iff [\mathbf{x}_0,\mathbf{x}_0] < 0,$$

and the third case does not occur. Thus the notion of positive and negative type eigenvectors defined by means of the quadratic numerical range coincides with the notion of $\mathcal{J}$-positive and $\mathcal{J}$-negative type eigenvectors from Definition 1.10.6.

***Proof.*** If we subtract (2.6.14) from (2.6.13), we arrive at

$$\left(\frac{1}{\|x_0\|^2} + \frac{1}{\|y_0\|^2}\right)(By_0, x_0) = -\frac{(Ax_0, x_0)}{\|x_0\|^2} + \frac{(Dy_0, y_0)}{\|y_0\|^2} < 0$$

and thus, by (2.6.12),

$$\lambda'(1)\,[\mathbf{x}_0, \mathbf{x}_0] = 2\,\mathrm{Re}\,(By_0, x_0) = 2\,(By_0, x_0) < 0. \qquad \square$$

To conclude this section, we apply Theorem 2.6.13 to a matrix differential operator considered already in Section 2.4, which is related to the $\lambda$-rational Sturm-Liouville problem (2.4.1).

**Example 2.6.15** In Example 2.4.3 we have studied the spectral problem

$$\mathcal{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\mathrm{D}^2 + q & \sqrt{|w|} \\ (\operatorname{sign} w)\sqrt{|w|} & u \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1(0) = y_1(1) = 0,$$

on $[0,1]$ under the assumptions $q \in L_1(0,1)$, $w, u \in L_\infty[0,1]$, and either $w \geq 0$ or $w \leq 0$. In Example 2.4.3 we have shown that $\sigma_{\mathrm{ess}}(\mathcal{A}) = u([0,1]) =: [u_-, u_+]$ if $u$ is continuous. If we assume that $u < 0$, *i.e.* $u_+ < 0$, then $\max\sigma(D) < 0 < \min\sigma(A)$. Then the spectrum $\sigma(\mathcal{A}) \cap (u_+, \infty)$ consists of isolated eigenvalues accumulating at $\infty$; the latter follows from the fact that $\mathcal{A}$ is bounded from below, but not from above.

a) $w \geq 0$: Then the corresponding block operator matrix $\mathcal{A}$ is self-adjoint in $L_2(0,1) \oplus L_2(0,1)$ by Theorem 2.6.6 i). Proposition 2.6.12 i) implies that all simple eigenvalues of $\mathcal{A}$ in $(u_+, \infty)$ are of positive type and thus, by Theorem 2.6.13, in a neighbourhood of $t = 1$, the corresponding eigenvalues of

$$\begin{pmatrix} -\mathrm{D}^2 + q & tw \\ tw & u \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1(0) = y_1(1) = 0, \qquad (2.6.15)$$

are monotonically increasing in $t$.

b) $w \leq 0$: Then the corresponding block operator matrix $\mathcal{A}$ is $\mathcal{J}$-self-adjoint in $L_2(0,1) \oplus L_2(0,1)$ by Theorem 2.6.6 ii). If we suppose that $\|w\| < \big(\min\sigma(A) - u_+\big)/2$, then $W^2(\mathcal{A}) \subset \mathbb{R}$ and $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$ for some $\gamma \in \big(u_+, \min\sigma(A)\big)$. Proposition 2.6.12 ii) implies that all simple eigenvalues of $\mathcal{A}$ in $(u_+, \infty)$ are of negative type and thus, by Theorem 2.6.13, in a neighbourhood of $t = 1$, the corresponding eigenvalues of

$$\begin{pmatrix} -\mathrm{D}^2 + q & tw \\ -tw & u \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1(0) = y_1(1) = 0, \qquad (2.6.16)$$

are monotonically decreasing in $t$.

The behaviour of eigenvalues under perturbations of $B$ can also be studied by means of the variational principles and eigenvalue estimates established in Sections 2.10 and 2.11.

## 2.7 Dichotomous block operator matrices and Riccati equations

In this section we study unbounded dichotomous block operator matrices. To ensure that no spectrum lies on the imaginary axis, we assume that the corners are symmetric, *i.e.* $C \subset B^*$, and that the spectra of the diagonal entries are separated by the imaginary axis. Our aim is to show that spectral subspaces corresponding to the spectrum in the open left and right half plane, respectively, exist and that they can be represented by means of so-called angular operators. As a consequence of the invariance of the spectral subspaces, the angular operators satisfy certain Riccati equations with unbounded operator coefficients.

In the bounded case, such angular operator representations have been proved in Theorem 1.7.1 under the assumption that the closure of the quadratic numerical range consists of two disjoint components. This condition is satisfied for dichotomous block operator matrices; in the unbounded case, however, the spectrum may still touch at $\infty$.

If the block operator matrix $\mathcal{A}$ is essentially self-adjoint, then the spectrum of $\mathcal{A}$ to the left and to the right of 0 can be separated at $\infty$ by means of the spectral projections of $\mathcal{A}$. For the non-self-adjoint case, we employ a theorem on the separation of the spectrum at $\infty$ of a closed linear operator due to H. Bart, I.C. Gohberg, and M.A. Kaashoek (see [BGK86] and Theorem 2.7.18 below). Here a stronger separation condition for the diagonal elements is required: in addition to being separated by the imaginary axis, the numerical ranges of the diagonal elements are assumed to lie in certain sectors with angle less than $\pi$. Moreover, the block operator matrix is supposed to be either diagonally dominant or off-diagonally dominant with some additional subordinacy properties; this guarantees that the Cauchy principal value of the integral of the resolvent of $\mathcal{A}$ along the imaginary axis exists.

In all cases, the key tool is a theorem on accretive linear operators in Krein spaces (see Theorem 2.7.5 below). It is a slight extension of a result of T.Ya. Azizov (see [AI89, Theorem II.2.21]); for dilating operators it was proved by I.S. Iokhvidov and M.G. Krein (see [IK56, Theorem 3.7] and also [IKL82, Theorem 11.1]).

For the following definitions and simple facts about Krein spaces and linear operators therein we refer to [AI89]. First we generalize the notions of $\mathcal{J}$-nonnegative *etc.* subspaces given in Definition 1.10.7 for $\mathcal{J} = \operatorname{diag}(I, -I)$ to more general operators $\mathcal{J}$.

**Definition 2.7.1** A linear operator $\mathcal{J}$ in the Hilbert space $\mathcal{H}$ is called a *self-adjoint involution* if $\mathcal{J}^* = \mathcal{J}$ and $\mathcal{J}^2 = I$ in $\mathcal{H}$.

**Remark 2.7.2** A linear operator $\mathcal{J}$ in $\mathcal{H}$ is a self-adjoint involution if and only if it is the difference of two complementary orthogonal projections $P_\pm$ :

$$\mathcal{J} = P_+ - P_-, \quad P_\pm^2 = P_\pm = P_\pm^*, \quad P_+ + P_- = I.$$

A self-adjoint involution generates an indefinite inner product $[\cdot,\cdot]_{\mathcal{J}}$ in $\mathcal{H}$ by

$$[x, y]_{\mathcal{J}} := (\mathcal{J}x, y), \quad x, y \in \mathcal{H}.$$

Equipped with $[\cdot,\cdot]_{\mathcal{J}}$, the space $\mathcal{H}$ becomes a Krein space. A particular case of a self-adjoint involution on $\mathcal{H}$ is the operator

$$\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

which was introduced in (1.1.10) (see also Section 1.10 and Section 2.6). All notions introduced there for this particular choice of $\mathcal{J}$, especially Definition 1.1.14 of $\mathcal{J}$-symmetric and $\mathcal{J}$-selfadjoint operators as well as Definition 1.10.7 of $\mathcal{J}$-nonnegative *etc.* subspaces carry over to arbitrary self-adjoint involutions $\mathcal{J}$ without change.

**Lemma 2.7.3** *If a Krein space is the direct sum of a $\mathcal{J}$-nonnegative subspace $\mathcal{L}_1$ and a $\mathcal{J}$-nonpositive subspace $\mathcal{L}_2$, then $\mathcal{L}_1$ is maximal $\mathcal{J}$-nonnegative andindexJ-nonpositive@$\mathcal{J}$-nonpositive!maximal $\mathcal{L}_2$ is maximal $\mathcal{J}$-nonpositive* (*see* [AI89, I.1.25°]).

**Definition 2.7.4** Let $\mathcal{J}$ be a self-adjoint involution in a Hilbert space $\mathcal{H}$ with associated inner product $[\cdot,\cdot]_{\mathcal{J}} = (\mathcal{J}\cdot,\cdot)$. A densely defined closed linear operator $T$ is called *$\mathcal{J}$-accretive* (*strictly $\mathcal{J}$-accretive*, respectively) if

$$\operatorname{Re}[Tx, x]_{\mathcal{J}} \geq 0 \quad (> 0, \text{ respectively}), \quad x \in \mathcal{D}(T),\ x \neq 0,$$

and *uniformly $\mathcal{J}$-accretive* if there exists a $\beta > 0$ such that

$$\operatorname{Re}[Tx, x]_{\mathcal{J}} \geq \beta \|x\|^2, \quad x \in \mathcal{D}(T). \tag{2.7.1}$$

In the following, the expressions under the integrals may have singularities not only at $\infty$, but also at 0. In both cases, $\int'$ denotes the Cauchy principal value of an integral at $\infty$, and possibly at 0.

**Theorem 2.7.5** *Let $\mathcal{J}$ be a self-adjoint involution in a Hilbert space $\mathcal{H}$ with associated inner product $[\cdot,\cdot]_{\mathcal{J}} = (\mathcal{J}\cdot,\cdot)$, let $P_\pm$ be the corresponding complementary orthogonal projections such that $\mathcal{J} = P_+ - P_-$, and define $\mathcal{K}_\pm := R(P_\pm)$ so that $\mathcal{H} = \mathcal{K}_+ \oplus \mathcal{K}_-$. Let $T$ be a $\mathcal{J}$-accretive linear operator in $\mathcal{H}$ such that $\mathrm{i}\mathbb{R}\setminus\{0\} \subset \rho(T)$ and the integral*

$$\frac{1}{\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}}' (T-z)^{-1} \mathrm{d}z$$

*exists in the strong operator topology. If there exist projections $Q_\pm$ with*

$$\frac{1}{\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}}' (T-z)^{-1} \mathrm{d}z = Q_+ - Q_-, \quad Q_+ + Q_- = I, \tag{2.7.2}$$

*then the subspaces $\mathcal{L}_\pm := R(Q_\pm)$ of $\mathcal{H}$ have the following properties:*

i) *$\mathcal{L}_+$ is maximal $\mathcal{J}$-nonnegative, $\mathcal{L}_-$ is maximal $\mathcal{J}$-nonpositive and hence there exist contractions $K_+ \in L(\mathcal{K}_+,\mathcal{K}_-)$, $K_- \in L(\mathcal{K}_-,\mathcal{K}_+)$ such that*

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x_+ \\ K_+ x_+ \end{pmatrix} : x_+ \in \mathcal{K}_+ \right\}, \quad \mathcal{L}_- = \left\{ \begin{pmatrix} K_- x_- \\ x_- \end{pmatrix} : x_- \in \mathcal{K}_- \right\}. \tag{2.7.3}$$

ii) *If $T$ is strictly $\mathcal{J}$-accretive (bounded and uniformly $\mathcal{J}$-accretive, respectively), then the subspace $\mathcal{L}_+$ is $\mathcal{J}$-positive (uniformly $\mathcal{J}$-positive, respectively), $\mathcal{L}_-$ is $\mathcal{J}$-negative (uniformly $\mathcal{J}$-negative, respectively), and the contractions $K_\pm$ in (2.7.3) are strict (uniform, respectively).*

***Proof.*** We prove the claims for the subspace $\mathcal{L}_+$; the proofs for $\mathcal{L}_-$ are analogous. Let $\mathbf{x} \in \mathcal{L}_+$, $\mathbf{x} \neq 0$, be arbitrary. Then $[\mathbf{x},\mathbf{x}]_{\mathcal{J}} = (\mathcal{J}\mathbf{x},\mathbf{x}) = (\mathbf{x},\mathcal{J}\mathbf{x}) = \overline{[\mathbf{x},\mathbf{x}]_{\mathcal{J}}}$ is real and we have $Q_+\mathbf{x} = \mathbf{x}$, $Q_-\mathbf{x} = 0$. Together with assumption (2.7.2), we obtain

$$\begin{aligned} [\mathbf{x},\mathbf{x}]_{\mathcal{J}} &= \operatorname{Re}[\mathbf{x},\mathbf{x}]_{\mathcal{J}} = \operatorname{Re}\big[Q_+\mathbf{x},\mathbf{x}\big]_{\mathcal{J}} = \operatorname{Re}\big[(Q_+ - Q_-)\mathbf{x},\mathbf{x}\big]_{\mathcal{J}} \\ &= \operatorname{Re}\left( \frac{1}{\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}}' \big[(T-z)^{-1}\mathbf{x},\mathbf{x}\big]_{\mathcal{J}} \,\mathrm{d}z \right) \\ &= \frac{1}{\pi} \int_{\mathbb{R}}' \operatorname{Re}\big[T(T-\mathrm{i}t)^{-1}\mathbf{x},(T-\mathrm{i}t)^{-1}\mathbf{x}\big]_{\mathcal{J}} \,\mathrm{d}t. \end{aligned} \tag{2.7.4}$$

If $T$ is $\mathcal{J}$-accretive (strictly $\mathcal{J}$-accretive, respectively), then the last integral is nonnegative (positive, respectively) and hence $\mathcal{L}_+$ is $\mathcal{J}$-nonnegative ($\mathcal{J}$-positive, respectively).

If $T$ is bounded and uniformly $\mathcal{J}$-accretive, we consider an arbitrary non-empty interval $(a,b) \subset \mathbb{R}$ with $a > 0$. Then, with $\beta$ as in (2.7.1), we can estimate the integral in (2.7.4) from below and arrive at

$$[\mathbf{x},\mathbf{x}]_{\mathcal{J}} \geq \frac{1}{\pi}\int_a^b \beta\,\big\|(T-\mathrm{i}t)^{-1}\mathbf{x}\big\|^2 \mathrm{d}t \geq \frac{\beta}{\pi}\,(b-a)\Big(\max_{t\in[a,b]}\|T-\mathrm{i}t\|\Big)^{-2}\|\mathbf{x}\|^2;$$

hence $\mathcal{L}_+$ is uniformly $\mathcal{J}$-positive.

That $\mathcal{L}_+$ is maximal $\mathcal{J}$-nonnegative is a consequence of the decomposition $\mathcal{K} = \mathcal{L}_+ \dotplus \mathcal{L}_-$ and Lemma 2.7.3. The angular operator representation of $\mathcal{L}_+$ and its properties claimed in i) and ii) follow from Remark 1.10.8. □

The next proposition shows that a block operator matrix $\mathcal{A}$ is $\mathcal{J}$-accretive with $\mathcal{J} = \operatorname{diag}(I, -I)$ if $A$ and $-D$ are accretive and $C \subset B^*$.

**Proposition 2.7.6** *Let $\mathcal{A}$ be a closable block operator matrix in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $C|_{\mathcal{D}(A)\cap\mathcal{D}(C)} \subset B^*$. If $A$ and $-D$ are accretive (uniformly accretive) in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ is $\mathcal{J}$-accretive (uniformly $\mathcal{J}$-accretive) with respect to the self-adjoint involution*

$$\mathcal{J} := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};$$

*if $\mathcal{A}$ is closed and $A$,$-D$ are strictly accretive, then $\mathcal{A}$ is strictly $\mathcal{J}$-accretive.*

***Proof.*** Let $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{D}(\mathcal{A})$. It is not difficult to see that, if $A$ and $-D$ are accretive, then

$$\operatorname{Re}[\mathcal{A}\mathbf{x},\mathbf{x}]_{\mathcal{J}} = \operatorname{Re}\big(\mathcal{J}\mathcal{A}\mathbf{x},\mathbf{x}\big) = \operatorname{Re}(Ax,x) - \operatorname{Re}(Dy,y) \geq 0. \tag{2.7.5}$$

Since $\overline{\mathcal{J}\mathcal{A}} = \mathcal{J}\overline{\mathcal{A}}$, this estimate continues to hold for the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$. The proof for the other cases is similar. □

In the following, we distinguish essentially self-adjoint block operator matrices and non-self-adjoint block operator matrices.

### 2.7.1 Essentially self-adjoint block operator matrices

The following theorem is the first main result of this section. It concerns the block diagonalizability of essentially self-adjoint block operator matrices. In contrast to the non-self-adjoint case considered later, no dominance assumptions on the entries are required here.

This theorem was first proved in [LT01, Theorem 3.1]. It generalizes earlier results by V.M. Adamjan and H. Langer in [AL95, Theorem 2.3] for bounded $B$ and $D$; in [ALMS96, Theorem 5.3] it was extended to upper dominant $\mathcal{A}$ with certain domain restrictions by V.M. Adamjan, H. Langer, R. Mennicken, and J. Saurer. The proofs of these results are different from the proof of [LT01, Theorem 3.1] given below; they do not make use of the $\mathcal{J}$-accretivity.

**Theorem 2.7.7** *Suppose that the block operator matrix $\mathcal{A}$ is essentially self-adjoint in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with closure $\overline{\mathcal{A}}$, the entries $A, B$, and $D$ are closed, and $0 \notin \sigma_{\mathrm{p}}(\overline{\mathcal{A}})$. Let $\mathcal{L}_+ := \mathcal{L}_{[0,\infty)}(\overline{\mathcal{A}})$, $\mathcal{L}_- := \mathcal{L}_{(-\infty,0]}(\overline{\mathcal{A}})$ be the spectral subspaces of $\overline{\mathcal{A}}$ corresponding to the intervals $[0,\infty)$ and $(-\infty,0]$, respectively (so that $\mathcal{H} = \mathcal{L}_+ \oplus \mathcal{L}_-$).*

i) *If the diagonal entries $A$ and $D$ satisfy*

$$\begin{aligned} (Ax,x) &\geq 0, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B^*), \\ (Dy,y) &\leq 0, \quad y \in \mathcal{D}(B) \cap \mathcal{D}(D), \end{aligned} \tag{2.7.6}$$

*then there exists a contraction $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ such that*

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_- = \left\{ \begin{pmatrix} -K^*y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\}. \tag{2.7.7}$$

ii) *If either $A$ and $D$ are bounded or if $B$ is bounded (so that $\mathcal{A}$ is self-adjoint), and if the inequalities (2.7.6) are strengthened to*

$$\begin{aligned} (Ax,x) &> 0, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B^*), \quad x \neq 0, \\ (Dy,y) &< 0, \quad y \in \mathcal{D}(B) \cap \mathcal{D}(D), \quad y \neq 0, \end{aligned} \tag{2.7.8}$$

*then the contraction $K$ in (2.7.7) is strict.*

iii) *If $B$ is bounded (so that $\mathcal{A}$ is self-adjoint) and the inequalities (2.7.6) are further strengthened to*

$$\begin{aligned} (Ax,x) &\geq \ \ \alpha \, \|x\|^2, \quad x \in \mathcal{D}(A), \\ (Dy,y) &\leq -\delta \, \|y\|^2, \quad y \in \mathcal{D}(D), \end{aligned} \tag{2.7.9}$$

*with some $\alpha$, $\delta > 0$, then the contraction $K$ in (2.7.7) is uniform and*

$$\big\{ z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha \big\} \subset \rho(\mathcal{A}).$$

***Proof.*** According to Proposition 2.7.6, the operator $\overline{\mathcal{A}}$ is $\mathcal{J}$-accretive with respect to the self-adjoint involution $\mathcal{J} = \operatorname{diag}(I, -I)$. Since $\overline{\mathcal{A}}$ is self-adjoint with respect to the Hilbert space inner product on $\mathcal{H}$, it satisfies all other assumptions of Theorem 2.7.5 with $Q_\pm$ denoting the orthogonal projections onto the spectral subspaces $\mathcal{L}_\pm$. Thus Theorem 2.7.5 applies and implies claims i) and ii) (note that $\mathcal{A}$ is closed in case ii)).

It remains to prove iii). For this, we show that there exists a $\gamma > 0$ with $[\mathbf{x},\mathbf{x}]_{\mathcal{J}} \geq \gamma \, \|\mathbf{x}\|^2$ for every $\mathbf{x} \in \mathcal{L}_+$. If $B$ is bounded, then $\mathcal{A} = \overline{\mathcal{A}}$ is closed and $A$, $D$ are self-adjoint on their domains. By (2.7.4), we have

$$\begin{aligned} [\mathbf{x},\mathbf{x}]_{\mathcal{J}} &= \frac{1}{\pi} \int_{\mathbb{R}}' \operatorname{Re} \big[ \mathcal{A}(\mathcal{A} - \mathrm{i}t)^{-1}\mathbf{x}, (\mathcal{A} - \mathrm{i}t)^{-1}\mathbf{x} \big] \, \mathrm{d}t \\ &\geq \frac{1}{\pi} \int_{|t| \geq t_0}' \operatorname{Re} \big[ \mathcal{A}(\mathcal{A} - \mathrm{i}t)^{-1}\mathbf{x}, (\mathcal{A} - \mathrm{i}t)^{-1}\mathbf{x} \big] \, \mathrm{d}t \end{aligned} \tag{2.7.10}$$

for arbitrary $t_0 \geq 0$ to be chosen later. Using (2.7.5) and the formulae for the resolvent of $\mathcal{A}$ in Theorem 2.3.3, the integrand in (2.7.10) can be written as

$$\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} (\mathcal{A} - \mathrm{i}t)^{-1}\mathbf{x}, (\mathcal{A} - \mathrm{i}t)^{-1}\mathbf{x}\right]_{\mathcal{J}} = \big(Au(t), u(t)\big) - \big(Dv(t), v(t)\big)$$

for $t \in \mathbb{R}$ where, for $\mathbf{x} = (x\ y)^{\mathrm{t}}$,

$$u(t) := S_1(\mathrm{i}t)^{-1}\big(x - B\ (D - \mathrm{i}t)^{-1}y\big),$$
$$v(t) := S_2(\mathrm{i}t)^{-1}\big(y - B^*(A - \mathrm{i}t)^{-1}x\big).$$

It is not difficult to see that $u(t)$ and $v(t)$ can be written as

$$\begin{aligned} u(t) &= \big((A- \mathrm{i}t)^{-1}+ (A- \mathrm{i}t)^{-1}B(D- \mathrm{i}t)^{-1}B^*S_1(\mathrm{i}t)^{-1}\big)\big(x-B(D- \mathrm{i}t)^{-1}y\big) \\ &=: (A - \mathrm{i}t)^{-1}x - (A - \mathrm{i}t)^{-1}B(D - \mathrm{i}t)^{-1}y + u_r(t), \qquad (2.7.11) \\ v(t) &= \big((D- \mathrm{i}t)^{-1}+ (D- \mathrm{i}t)^{-1}B^*(A- \mathrm{i}t)^{-1}BS_2(\mathrm{i}t)^{-1}\big)\ \big(y-B^*(A- \mathrm{i}t)^{-1}x\big) \\ &=: (D - \mathrm{i}t)^{-1}y - (D - \mathrm{i}t)^{-1}B^*(A - \mathrm{i}t)^{-1}x + v_r(t), \qquad (2.7.12) \end{aligned}$$

where $u_r(t)$, $v_r(t)$ consist of the terms containing at least three inverses. In order to estimate the integral in (2.7.10) from below, we split

$$\begin{aligned} &\int_{t_0}^{\infty} \big(Au(t), u(t)\big) - \big(Dv(t), v(t)\big)\,\mathrm{d}t \\ &=: a(t_0, x) + d(t_0, y) + a_1(t_0, x, y) + d_1(t_0, x, y) + r(t_0, x, y) \end{aligned}$$

where the leading terms $a(t_0, x)$, $d(t_0, y)$ contain only two inverses, the terms $a_1(t_0, x, y)$, $d_1(t_0, x, y)$ are sums of two complex conjugate terms containing three inverses, and the term $r(t_0, x, y)$ contains all remaining terms involving at least four inverses, one of them being the product $A(A - \mathrm{i}t)^{-1}$ or $D(D - \mathrm{i}t)^{-1}$.

Using the spectral theorem for the self-adjoint operator $A$, we obtain

$$\begin{aligned} a(t_0, x) &= \int_{t_0}^{\infty} \big(A(A - \mathrm{i}t)^{-1}x, (A - \mathrm{i}t)^{-1}x\big)\,\mathrm{d}t \\ &\geq \left(\frac{\pi}{2} - \arctan\frac{t_0}{\alpha}\right)\|x\|^2 =: \gamma_1^{\alpha}(t_0)\,\|x\|^2, \qquad (2.7.13) \end{aligned}$$

where $\gamma_1^{\alpha}(t_0) > 0$ and $\gamma_1^{\alpha}(t_0) = \mathrm{O}(t_0^{-1})$ as $t_0 \to \infty$. A completely analogous estimate, with a constant $\gamma_1^{\delta}(t_0)$, can be proved for $d(t_0, y)$.

In order to estimate the integral over the mixed terms involving three inverses, we use the inequalities

$$\big\|(A + \mathrm{i}t)^{-1}A(A - \mathrm{i}t)^{-1}\big\| \leq \frac{1}{2t}, \quad t > 0,$$
$$\int_{t_0}^{\infty} \frac{1}{t}\,\big\|(D - \mathrm{i}t)^{-1}y\big\|^2\,\mathrm{d}t \leq \frac{1}{\delta^2}\ln\frac{\sqrt{t_0^2 + \delta^2}}{t_0}\,\|y\|^2,$$

which follow by means of the spectral theorem for the self-adjoint operators $A$ and $D$. Together with the Cauchy–Schwarz inequality with respect to the scalar product $\big(A(A-\mathrm{i}t)^{-1}\cdot,(A-\mathrm{i}t)^{-1}\cdot\big)$, we thus obtain

$$\begin{aligned}
\big|a_1(t_0,x,y)\big| &= \left|2\,\mathrm{Re}\int_{t_0}^{\infty}\big(A(A-\mathrm{i}t)^{-1}x,(A-\mathrm{i}t)^{-1}B(D-\mathrm{i}t)^{-1}y\big)\,\mathrm{d}t\right| \\
&\le 2\,a(t_0,x)^{1/2}\left(\frac{\|B\|^2}{2}\int_{t_0}^{\infty}\frac{1}{t}\,\big\|(D-\mathrm{i}t)^{-1}y\big\|^2\mathrm{d}t\right)^{1/2} \\
&\le a(t_0,x)^{1/2}\,\frac{\sqrt{2}\,\|B\|}{\delta}\left(\ln\frac{\sqrt{t_0^2+\delta^2}}{t_0}\right)^{1/2}\|y\| \\
&=: a(t_0,x)^{1/2}\,\gamma_2^{\alpha}(t_0)\,\|y\| \\
&\le \frac{1}{2}\Big(\frac{1}{2}a(t_0,x)+2\big(\gamma_2^{\alpha}(t_0)\big)^2\,\|y\|^2\Big).
\end{aligned} \tag{2.7.14}$$

Again, a completely analogous estimate, with a constant $\gamma_2^{\delta}(t_0)$, can be proved for $d_1(t_0,x,y)$.

Altogether, we arrive at the estimate

$$\begin{aligned}
&\frac{1}{2}\big(a(t_0,x)+d(t_0,y)\big)+a_1(t_0,x,y)+d_1(t_0,x,y) \\
&\ge \frac{1}{4}\big(a(t_0,x)+d(t_0,y)\big)-\big(\gamma_2^{\alpha}(t_0)\big)^2\,\|x\|^2-\big(\gamma_2^{\delta}(t_0)\big)^2\,\|y\|^2 \\
&\ge \frac{1}{4}a(t_0,x)\Big(1-\frac{\big(\gamma_2^{\alpha}(t_0)\big)^2\,\|x\|^2}{a(t_0,x)}\Big)+\frac{1}{4}d(t_0,x)\Big(1-\frac{\big(\gamma_2^{\delta}(t_0)\big)^2\,\|y\|^2}{d(t_0,y)}\Big) \\
&\ge \frac{1}{4}\gamma_1^{\alpha}(t_0)\Big(1-\frac{\big(\gamma_2^{\alpha}(t_0)\big)^2}{\gamma_1^{\alpha}(t_0)}\Big)\,\|x\|^2+\frac{1}{4}\gamma_1^{\delta}(t_0)\Big(1-\frac{\big(\gamma_2^{\delta}(t_0)\big)^2}{\gamma_1^{\delta}(t_0)}\Big)\,\|y\|^2.
\end{aligned}$$

By the choice of $\gamma_1^{\delta}(t_0)$, we have $\gamma_1^{\delta}(t_0)>0$. Moreover, it is not difficult to see that *e.g.* $\big(\gamma_2^{\alpha}(t_0)\big)^2\!/\gamma_1^{\alpha}(t_0)\to 0$ for $t_0\to\infty$. Together with analogous considerations for the other term, we can choose $t_1>0$ so that

$$\frac{1}{2}\big(a(t_0,x)+d(t_0,y)\big)+a_1(t_0,x,y)+d_1(t_0,x,y)\ge\gamma^{\alpha}(t_0)\,\|x\|^2+\gamma^{\delta}(t_0)\,\|y\|^2$$

with constants $\gamma^{\alpha}(t_0)>0$, $\gamma^{\delta}(t_0)>0$ for all $t_0\ge t_1$.

The remaining term $r(t_0,x,y)$ can be estimated simply using the norm estimates $\big\|A(A-\mathrm{i}t)^{-1}\big\|\le 1$, $\big\|(A-\mathrm{i}t)^{-1}\big\|\le t^{-1}$, $\big\|(D-\mathrm{i}t)^{-1}\big\|\le t^{-1}$, and $\big\|S_1(\mathrm{i}t)^{-1}\big\|\le\kappa\,t^{-1}$ with some $\kappa>0$ for $t>0$. This implies that the absolute value of $r(t_0,x,y)$ is bounded from above by

$$c_1(t_0)\,\|x\|^2,\quad c_2(t_0)\,\|x\|\,\|y\|,\quad\text{or}\quad c_3(t_0)\,\|y\|^2$$

with $c_i(t_0)=\mathrm{O}(t_0^{-2})$ for $i=1,2,3$. In a similar way as above, it can be shown that we can choose $\widetilde{t_1}\ge t_1>0$ such that

$$\frac{1}{2}\big(a(t_0,x)+d(t_0,y)\big)+r(t_0,x,y)\geq \widetilde{\gamma}^{\alpha}(t_0)\,\|x\|^2+\widetilde{\gamma}^{\delta}(t_0)\,\|y\|^2$$

with constants $\widetilde{\gamma}^{\alpha}(t_0)>0$, $\widetilde{\gamma}^{\delta}(t_0)>0$ for all $t_0\geq \widetilde{t}_1\geq t_1$. This completes the proof that $\mathcal{L}_+$ is uniformly positive.

The very last claim follows from Theorem 2.5.18 and from the fact that $\mathrm{i}\mathbb{R}\setminus\{0\}\subset\rho(\mathcal{A})$ since $\mathcal{A}$ is self-adjoint. □

The invariance of the spectral subspaces $\mathcal{L}_\pm$ under $\overline{\mathcal{A}}$ implies that the angular operator $K$ satisfies a Riccati equation (see Theorem 1.7.1 for the bounded case). In the unbounded case, however, the Riccati equations can, in general, not be written in the form $KBK+KA-DK-B^*=0$. For instance, the Riccati equation may not hold on all of $\mathcal{D}(A)\cap\mathcal{D}(B^*)$ due to domain restrictions in the terms $KBK$ and $DK$, and it may have to be written differently if $\mathcal{A}$ is not closed.

**Corollary 2.7.8** *Define $\mathcal{D}_+\subset\mathcal{H}_1$ and $\mathcal{D}_-\subset\mathcal{H}_2$ by*

$$\mathcal{D}_+:=\left\{x\in\mathcal{H}_1:\begin{pmatrix}x\\Kx\end{pmatrix}\in\mathcal{D}\big(\overline{\mathcal{A}}\big)\right\},\ \mathcal{D}_-:=\left\{y\in\mathcal{H}_2:\begin{pmatrix}-K^*y\\y\end{pmatrix}\in\mathcal{D}\big(\overline{\mathcal{A}}\big)\right\},\tag{2.7.15}$$

*and let the Schur complements $S_1$, $S_2$ and quadratic complements $T_1$, $T_2$ be defined as in* (2.2.8), (2.2.9) *and* (2.2.14), (2.2.13), *respectively, with $C=B^*$.*

*Then the angular operator $K$ in Theorem* 2.7.7 *and its adjoint $K^*$ satisfy Riccati equations of the following form on $\mathcal{D}_+$ and $\mathcal{D}_-$, respectively:*

i) *if $\mathcal{D}(A)\subset\mathcal{D}(B^*)$, $\rho(A)\neq\emptyset$, and for some (and hence for all) $\mu\in\rho(A)$ the operator $(A-\mu)^{-1}B$ is bounded on $\mathcal{D}(B)$, then*

$$K(A-\mu)\big(\overline{(A-\mu)^{-1}B}K+I\big)-\overline{S_2(\mu)}K-B^*\big(\overline{(A-\mu)^{-1}B}K+I\big)=0;$$

ii) *if $\mathcal{D}(B^*)\subset\mathcal{D}(A)$, $B^*$ is boundedly invertible, and for some (and hence for all) $\mu\in\mathbb{C}$ the operator $B^{-*}(D-\mu)$ is bounded on $\mathcal{D}(D)$, then*

$$K\overline{T_2(\mu)}K+K(A-\mu)\big(I+\overline{B^{-*}(D-\mu)}\big)-B^*\big(I+\overline{B^{-*}(D-\mu)}\big)=0;$$

iii) *if $\mathcal{D}(D)\subset\mathcal{D}(B)$, $\rho(D)\neq\emptyset$, and for some (and hence for all) $\mu\in\rho(D)$ the operator $(D-\mu)^{-1}B^*$ is bounded on $\mathcal{D}(B^*)$, then*

$$KB\big(K+\overline{(D-\mu)^{-1}B^*}\big)+K\overline{S_1(\mu)}-(D-\mu)\big(K+\overline{(D-\mu)^{-1}B^*}\big)=0;$$

iv) *if $\mathcal{D}(B)\subset\mathcal{D}(D)$, $B$ is boundedly invertible, and for some (and hence for all) $\mu\in\mathbb{C}$ the operator $B^{-1}(A-\mu)$ is bounded on $\mathcal{D}(A)$, then*

$$KB\big(K+\overline{B^{-1}(A-\mu)}\big)-(D-\mu)\big(K+\overline{B^{-1}(A-\mu)}\big)-\overline{T_1(\mu)}-\mu K=0;$$

*the Riccati equations for $K^*$ are analogous.*

***Proof.*** All Riccati equations above are obtained from the invariance of $\mathcal{L}_+$ under $\overline{\mathcal{A}}$, that is, $\overline{\mathcal{A}}\big(\mathcal{L}_+ \cap \mathcal{D}(\overline{\mathcal{A}})\big) \subset \mathcal{L}_+$, together with the angular operator representation (2.7.7) of $\mathcal{L}_+$ and the different formulae for the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in Theorems 2.2.14, 2.2.23, 2.2.18, and 2.2.25. □

**Remark 2.7.9** On the set

$$\mathcal{D}_{0,+} := \left\{ x \in \mathcal{H}_1 : \begin{pmatrix} x \\ Kx \end{pmatrix} \in \mathcal{D}(\mathcal{A}) \right\} \subset \mathcal{D}_+,$$

all the Riccati equations in Corollary 2.7.8 reduce to the standard form $KBK + KA - DK - B^* = 0$; note that $\mathcal{D}_{0,+} = \mathcal{D}_+$ if $\mathcal{A}$ is self-adjoint (and hence closed).

**Remark 2.7.10** The set $\mathcal{D}_+$ is the first component of $\mathcal{L}_+ \cap \mathcal{D}(\overline{\mathcal{A}})$. Thus two conditions are imposed on $x \in \mathcal{D}_+$, *e.g.* $\big(\overline{(A-\mu)^{-1}B}K + I\big)x \in \mathcal{D}(A)$ and $Kx \in \mathcal{D}\big(\overline{S_2(\mu)}\big)$ in case i). In general, it is not clear whether the first condition implies the second.

The set $\mathcal{D}_{0,+}$ is the first component of $\mathcal{L}_+ \cap \mathcal{D}(\mathcal{A})$; here the two conditions imposed on $x \in \mathcal{D}_{0,+}$ are $x \in \mathcal{D}(A) \cap \mathcal{D}(B^*)$ and $Kx \in \mathcal{D}(B) \cap \mathcal{D}(D)$. Here the first condition would imply the second if $K\big(\mathcal{D}(A) \cap \mathcal{D}(B^*)\big) \subset \mathcal{D}(B) \cap \mathcal{D}(D)$. In this case, the Riccati equation would hold on the whole first component $\mathcal{D}_1 = \mathcal{D}(A) \cap \mathcal{D}(B^*)$ of the domain of $\mathcal{A}$. For diagonally dominant block operator matrices, this will be shown in Corollary 2.7.23.

The condition $0 \notin \sigma_{\mathrm{p}}\big(\overline{\mathcal{A}}\big)$ in Theorem 2.7.7 may not be satisfied if the spectra of the diagonal elements $A$ and $D$ touch at 0. However, Theorem 2.7.7 and Corollary 2.7.8 continue to hold if $0 \in \sigma_{\mathrm{p}}\big(\overline{\mathcal{A}}\big)$ provided that the kernel of $\overline{\mathcal{A}}$ admits a suitable decomposition:

**Definition 2.7.11** For a closable block operator matrix $\mathcal{A}$ in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with closure $\overline{\mathcal{A}}$, the kernel of $\overline{\mathcal{A}}$ is said to have the *kernel splitting property* if

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \ker \overline{\mathcal{A}} \quad \Longrightarrow \quad \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \in \ker \overline{\mathcal{A}}, \ \begin{pmatrix} 0 \\ y_0 \end{pmatrix} \in \ker \overline{\mathcal{A}}. \tag{2.7.16}$$

**Remark 2.7.12** If $0 \in \sigma_{\mathrm{p}}\big(\overline{\mathcal{A}}\big)$ and $\ker \overline{\mathcal{A}}$ has the kernel splitting property, then all claims of Theorem 2.7.7 continue to hold with

$$\mathcal{L}_+ := \mathcal{L}_{(0,\infty)}\big(\overline{\mathcal{A}}\big) \dotplus \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{H} : \exists\, y \in \mathcal{H}_2 \begin{pmatrix} x \\ y \end{pmatrix} \in \ker \overline{\mathcal{A}} \right\},$$

and analogously for $\mathcal{L}_-$.

The following proposition contains sufficient criteria for the kernel splitting property for block operator matrices as considered in Theorem 2.7.7.

**Proposition 2.7.13** *Let $\mathcal{A}$ be a closable block operator matrix in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with closure $\overline{\mathcal{A}}$. Suppose that $A$ and $-D$ are regularly m-accretive and that $B$ is closed. Then*

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \ker \overline{\mathcal{A}} \quad \Longrightarrow x_0 \in \ker A, \;\; y_0 \in \ker D.$$

*If, in addition, one of the conditions*

(i) *$\mathcal{A}$ is closed,*
(ii) *$\mathcal{A}$ is diagonally dominant,*
(iii) *at least one of $A$ and $D$ is bounded,*

*holds, then* $\ker \overline{\mathcal{A}}$ *has the kernel splitting property. If* (i) *or* (ii) *hold, then*

$$\ker \overline{\mathcal{A}} = \left\{ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathcal{D}(\mathcal{A}) : \; Ax_0 = By_0 = 0, \; Dy_0 = B^* x_0 = 0 \right\};$$

*in the particular case of* (i) *that $A$ and $D$ are bounded or that $B$ is bounded, we have* $\ker \mathcal{A}^* = \ker \mathcal{A}$.

For the proof of Proposition 2.7.13, we need some properties of regularly m-accretive operators; for the definition of m-accretive operators we refer to Section 2.1 (see Definition 2.1.17). The following theorem and its proof may be found in [Kat95, Theorem VI.3.2]; recall that therein regularly m-accretive operators are called m-sectorial with vertex 0 (see Remark 2.1.23).

**Theorem 2.7.14** *Let $T$ be a regularly m-accretive operator in a Hilbert space $\mathcal{H}$ with angle $\theta$. Then there exist a nonnegative operator $G$ in $\mathcal{H}$ and a self-adjoint operator $B \in L(\mathcal{H})$ with $\|B\| \le \tan\theta$ such that*

$$T = G(I + \mathrm{i}B)G. \tag{2.7.17}$$

The proof of this theorem relies on the fact that, for a regularly m-accretive operator, the real part $\operatorname{Re} T$ is defined and is a nonnegative operator; in this case one can choose $G = (\operatorname{Re} T)^{1/2}$. Since it is known that $\operatorname{Re} T = \operatorname{Re} T^*$, the following result is a consequence of Theorem 2.7.14; in fact, this is a particular case of the more general theorem that the purely imaginary eigenvalues of an m-accretive operator $T$ and its adjoint coincide (see [SNF70, Proposition IV.4.3]).

**Corollary 2.7.15** *If $T$ is a regularly m-accretive operator in a Hilbert space $\mathcal{H}$, then* $\ker T = \ker T^*$.

The above Theorem 2.7.14 can also be used to prove that if $0 \in W(T)$ for a regularly m-accretive operator $T$ (so that 0 is a corner of $W(T)$), then $0 \in \sigma_{\mathrm{p}}(T)$, and more generally, that every corner $\lambda_0 \in W(T)$ of the numerical range of a closed linear operator is an eigenvalue. We need the following stronger result.

**Lemma 2.7.16** *Let $T$ be a regularly m-accretive operator in a Hilbert space $\mathcal{H}$. If $(x_n)_1^\infty \subset \mathcal{D}(T)$, $x_0 \in \mathcal{H}$ are such that $x_n \to x_0$ and $(Tx_n, x_n) \to 0$, $n \to \infty$, then $x_0 \in \ker T$.*

***Proof.*** Let the operators $G$ and $B$ be as in Theorem 2.7.14. Then

$$\big((I + \mathrm{i}B)Gx_n, Gx_n\big) = \|Gx_n\|^2 + \mathrm{i}\big(BGx_n, Gx_n\big) \longrightarrow 0, \quad n \to \infty.$$

Since $B$ is self-adjoint, this implies $\|Gx_n\| \to 0$, $n \to \infty$. Because $x_n \to x_0$, $n \to \infty$, and $G$ is closed, we obtain $x_0 \in \mathcal{D}(G)$ and $Gx_0 = 0$. This implies $Gx_0 \in \mathcal{D}\big(G(I + \mathrm{i}B)\big)$, *i.e.* $x_0 \in \mathcal{D}(T)$, and $Tx_0 = 0$ according to (2.7.17).□

***Proof of Proposition* 2.7.13.** Let $(x_0\, y_0)^{\mathrm{t}} \in \ker \overline{\mathcal{A}}$. Then there exist sequences $(x_n)_1^\infty \subset \big(\mathcal{D}(A) \cap \mathcal{D}(B^*)\big)$, $(y_n)_1^\infty \subset \big(\mathcal{D}(B) \cap \mathcal{D}(D)\big)$ such that $x_n \to x_0$, $y_n \to y_0$, $n \to \infty$, and

$$Ax_n + By_n \longrightarrow 0, \quad B^*x_n + Dy_n \longrightarrow 0, \quad n \to \infty. \tag{2.7.18}$$

It follows that $(Ax_n, x_n) + (By_n, x_n) \to 0$, $(B^*x_n, y_n) + (Dy_n, y_n) \to 0$ and hence $(Ax_n, x_n) - \overline{(Dy_n, y_n)} \to 0$, $n \to \infty$. Since $A$ and $-D$ are accretive, this implies $(Ax_n, x_n) \to 0$, $(Dy_n, y_n) \to 0$, $n \to \infty$. Now Lemma 2.7.16 shows that $x_0 \in \ker A$ and $y_0 \in \ker D$.

If (i) holds, then $\mathcal{A} = \overline{\mathcal{A}}$ and so $(x_0\ y_0)^{\mathrm{t}} \in \ker \overline{\mathcal{A}} = \ker \mathcal{A}$; if (ii) holds, then, by what was shown above, $x_0 \in \mathcal{D}(A) \subset \mathcal{D}(B^*)$, $y_0 \in \mathcal{D}(D) \subset \mathcal{D}(B)$ and thus $(x_0\ y_0)^{\mathrm{t}} \in \ker \mathcal{A}$. Together with $x_0 \in \ker A$, $y_0 \in \ker D$, we find

$$\begin{pmatrix} By_0 \\ B^*x_0 \end{pmatrix} = \mathcal{A}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0.$$

This shows that $(x_0\, 0)^{\mathrm{t}}, (0\, y_0)^{\mathrm{t}} \in \ker \mathcal{A}$. Hence $\ker \overline{\mathcal{A}}$ has the kernel splitting property and possesses the form claimed in the theorem.

If $A$ is bounded, then $Ax_n \to Ax_0 = 0$ and the first relation in (2.7.18) yields $By_n \to 0$, $n \to \infty$. Since $B$ is closed, this implies $y_0 \in \mathcal{D}(B)$ and $By_0 = 0$. It follows that $(0\ y_0)^{\mathrm{t}} \in \ker \mathcal{A} \subset \ker \overline{\mathcal{A}}$ and hence also $(x_0\ 0)^{\mathrm{t}} \in \ker \overline{\mathcal{A}}$. Analogously, if $D$ is bounded, we obtain $(x_0\ 0)^{\mathrm{t}} \in \ker \mathcal{A} \subset \ker \overline{\mathcal{A}}$. Hence also in this case, $\ker \overline{\mathcal{A}}$ has the kernel splitting property.

If either $A$ and $D$ are bounded or if $B$ is bounded, then $\mathcal{A}$ is closed, and, by Proposition 2.6.3, the adjoint of $\mathcal{A} = \overline{\mathcal{A}}$ is given by

$$\mathcal{A}^* = \begin{pmatrix} A^* & B \\ B^* & D^* \end{pmatrix}.$$

The representation for $\ker \mathcal{A}^*$ follows if we apply the above to $\mathcal{A}^*$ and observe that, since $A$ and $-D$ are regularly m-accretive, so are $A^*$ and $-D^*$ and we have $\ker A = \ker A^*$ and $\ker D = \ker D^*$ by Corollary 2.7.15. □

**Remark 2.7.17** Analogues of Theorem 2.7.7 and Corollary 2.7.8 were proved in some other cases under partly more general, but partly more restrictive conditions on the entries:

For upper dominant essentially self-adjoint block operator matrices with bounded $D$, the separation condition $\max \sigma(D) \le \min \sigma(A)$ was weakened by R. Mennicken and A.A. Shkalikov (see [MS96, Theorem 2.5]). They only assumed that $A$ is bounded from below and that, for some $\gamma < \min \sigma(A)$, the Schur complement $S_2(\gamma)$ is uniformly negative; this allowed them to apply factorization theorems for the Schur complement $S_2$.

Essentially $\mathcal{J}$-self-adjoint block operator matrices

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$$

with $\max \sigma(D) < \min \sigma(A)$ were first considered in [AL95, Theorem 4.1] for bounded $B$, and later in [MS96, Theorem 3.2] for bounded $D$. In [AL95] it was assumed that $|(By,x)| \le \big((A-\gamma)x,x\big)\big((\gamma-D)y,y\big)$, $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(D)$, while in [MS96] it was supposed that $S_2(\gamma)$ is uniformly negative for some $\beta \in \big(\max \sigma(D), \min \sigma(A)\big)$. Both conditions, in fact, guarantee that the quadratic numerical range of $\mathcal{A}$ is real (compare Proposition 2.6.10 and its proof). Note that here the spectral subspaces $\mathcal{L}_+$, $\mathcal{L}_-$ are $\mathcal{J}$-orthogonal so that in the angular operator representation (2.7.7) of $\mathcal{L}_-$ we have $K^*$ instead of $-K^*$ and in Corollary 2.7.8 one has to replace $B^*$ by $-B^*$.

### 2.7.2 Non-self-adjoint block operator matrices

In the remaining part of this section, we consider non-self-adjoint diagonally dominant and off-diagonally dominant block operator matrices. Here the subspaces $\mathcal{L}_\pm$ cannot be defined by means of the spectral projections of $\mathcal{A}$. Additional assumptions are needed to guarantee that the spectrum separates at $\infty$ and assumption (2.7.2) of Theorem 2.7.5 is satisfied.

The theorem below is a consequence of the condition for the splitting of the spectrum at $\infty$ for a dichotomous operator which was established by H. Bart, I.C. Gohberg, and M.A. Kaashoek (see [BGK86], [GGK90, Theorem XV.3.1]); other conditions for this splitting, in terms of powers of the operator, were given in [DV89]. Here and in the sequel, $\mathbb{C}_+$, $\mathbb{C}_-$ denote the open right and left half plane, respectively.

**Theorem 2.7.18** *Let $E$ be a Banach space, $T$ a densely defined closed linear operator in $E$, and suppose that there exists an $h > 0$ such that the following conditions are satisfied:*

(i) $\{z \in \mathbb{C} : -h < \operatorname{Re} z < h\} \subset \rho(T)$,

(ii) $\displaystyle\sup_{-h<\operatorname{Re} z<h} \|(T-z)^{-1}\| < \infty$,

(iii) $\displaystyle\lim_{\eta\to\infty} \sup \left\{ \|(T-z)^{-1}\| : z = \xi \pm \mathrm{i}\eta, 0 \le \xi < h \right\} = 0$,

(iv) $\displaystyle\frac{1}{\pi\mathrm{i}} \int'_{\mathrm{i}\mathbb{R}} (T-z)^{-1} \mathrm{d}z$ *exists in the strong operator topology,*

*where $\int'$ denotes the Cauchy principal value at $\infty$. Then the integral in* (iv) *is the difference of two complementary projections, that is, there exist projections $Q_+, Q_-$ in $E$ such that*

$$\frac{1}{\pi\mathrm{i}} \int'_{\mathrm{i}\mathbb{R}} (T-z)^{-1} \mathrm{d}z = Q_+ - Q_-, \quad Q_+ + Q_- = I. \tag{2.7.19}$$

*If we let $\mathcal{X}_\pm := R(Q_\pm)$, then the sets $\mathcal{X}_\pm \cap \mathcal{D}(T)$ are dense in $\mathcal{X}_\pm$ and*

$$\mathcal{X}_\pm \cap \mathcal{D}(T) = Q_\pm \mathcal{D}(T), \quad T\big(\mathcal{X}_\pm \cap \mathcal{D}(T)\big) = \mathcal{X}_\pm, \quad \sigma\big(T|_{\mathcal{X}_\pm}\big) = \sigma(T) \cap \mathbb{C}_\pm.$$

***Proof.*** In view of [GGK90, Theorem XV.3.1], we have to prove that, for arbitrary $\alpha \in (0, h)$, the integral

$$\frac{1}{2\pi\mathrm{i}} \int_{\alpha+\mathrm{i}\mathbb{R}} z^{-2} (T-z)^{-1} T^2 x \,\mathrm{d}z$$

defines a bounded linear operator $Q_+ \in L(E)$. To this end, let $\alpha \in (0, h)$. Assumption (i) and Cauchy's theorem show that, for $t_0 > 0$,

$$\begin{aligned} \frac{1}{2\pi} \int_{-t_0}^{t_0} (T - \mathrm{i}t)^{-1} \mathrm{d}t = {}& \frac{1}{2\pi} \int_{-t_0}^{t_0} (T - (\alpha + \mathrm{i}t))^{-1} \mathrm{d}t \\ & - \frac{1}{2\pi\mathrm{i}} \left( \int_0^\alpha (T - (s + \mathrm{i}t_0))^{-1} \mathrm{d}s - \int_\alpha^0 (T - (s - \mathrm{i}t_0))^{-1} \mathrm{d}s \right). \end{aligned}$$

If we let $t_0 \to \infty$, then the last two integrals tend to 0 by assumption (ii), and the limit of the integral on the left hand side exists in the strong operator topology by assumption (iv). Thus also the limit of the first integral on the right hand side exists in the strong operator topology and

$$\frac{1}{2\pi\mathrm{i}} \int'_{\mathrm{i}\mathbb{R}} (T-z)^{-1} \mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int'_{\alpha+\mathrm{i}\mathbb{R}} (T-z)^{-1} \mathrm{d}t.$$

Using the relation $z^{-2}T^2 = I + z^{-2}(T+z)(T-z)$, we find that, for $x \in \mathcal{D}(T^2)$,

$$\frac{1}{2\pi \mathrm{i}}\int_{\alpha+\mathrm{i}\mathbb{R}} z^{-2}(T-z)^{-1}T^2x\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int'_{\alpha+\mathrm{i}\mathbb{R}}(T-z)^{-1}\mathrm{d}z\,x + \frac{1}{2\pi\mathrm{i}}\int_{\alpha+\mathrm{i}\mathbb{R}} z^{-2}\mathrm{d}z\;Tx + \frac{1}{2\pi\mathrm{i}}\int'_{\alpha+\mathrm{i}\mathbb{R}} z^{-1}\mathrm{d}z\;x;$$

here, due to the residue theorem, the second integral vanishes and the third integral equals $x/2$. Altogether, it follows that the integral on the left hand side defines a bounded linear operator $Q_+ \in L(E)$ and, with $Q_- := I - Q_+$,

$$Q_+ = \frac{1}{2\pi\mathrm{i}}\int'_{\alpha+\mathrm{i}\mathbb{R}}(T-z)^{-1}\mathrm{d}z + \frac{1}{2}I = \frac{1}{2\pi\mathrm{i}}\int'_{\mathrm{i}\mathbb{R}}(T-z)^{-1}\mathrm{d}z + \frac{1}{2}(Q_+ + Q_-),$$

which proves (2.7.19). That $Q_+$ and $Q_-$ are projections with the claimed properties follows from [GGK90, Theorem XV.3.1]. □

The assumptions of Theorem 2.7.18 will be satisfied for the dominating part of the non-self-adjoint block operator matrices considered later. To ensure that this continues to hold for the whole block operator matrix, the following perturbation result is used.

**Theorem 2.7.19** *Let $E$ be a Banach space and let $T$ be a densely defined closed linear operator in $E$ fulfilling the conditions* (i), (ii), (iii), *and* (iv) *of Theorem* 2.7.18. *Let $S$ be a linear operator in $E$ with $\mathcal{D}(T) \subset \mathcal{D}(S)$ and such that there exist $\gamma$, $\gamma' \geq 0$ and $\eta > 0$ with*

(v) $\left\|S(T-\mathrm{i}t)^{-1}\right\| \leq \dfrac{\gamma}{1+|t|^\eta}, \quad \left\|(T+S-\mathrm{i}t)^{-1}\right\| \leq \dfrac{\gamma'}{1+|t|}, \quad t \in \mathbb{R}.$

*Then the operator $T + S$ is closed, and if there exists $h' > 0$ such that $\{z \in \mathbb{C} : -h' < \operatorname{Re} z < h'\} \subset \rho(T+S)$, then*

$$\frac{1}{\pi\mathrm{i}}\int'_{\mathrm{i}\mathbb{R}}(T+S-z)^{-1}\mathrm{d}z = Q'_+ - Q'_-, \quad Q'_+ + Q'_- = I$$

*with two projections $Q'_\pm$ having the same properties with respect to $T+S$ as the projections $Q_\pm$ in Theorem* 2.7.18 *have with respect to $T$.*

***Proof.*** We show that the operator $T + S$ satisfies the same conditions as $T$ with $h'$ instead of $h$. By assumption, this is true for condition (i) of Theorem 2.7.18. For condition (ii), we observe that, for $z \in \rho(T) \cap \rho(T+S)$,

$$(T+S-z)^{-1} = (T-z)^{-1}\left(I + S(T-z)^{-1}\right)^{-1}$$

and

$$\begin{aligned} S(T-z)^{-1} &= S\big((T-z)^{-1} - (T-\mathrm{i}\operatorname{Im} z)^{-1}\big) + S(T-\mathrm{i}\operatorname{Im} z)^{-1} \\ &= S(T-\mathrm{i}\operatorname{Im} z)^{-1}\big(\operatorname{Re} z\,(T-z)^{-1} + I\big). \end{aligned}$$

Since $M := \sup_{-h<\operatorname{Re} z<h} \big\|(T-z)^{-1}\big\| < \infty$ by assumption (ii) of Theorem 2.7.18 for $T$, we can choose $t_0 > 0$ sufficiently large so that

$$\frac{\gamma}{1+t_0^\eta}\big(h'M+1\big) < \frac{1}{2}$$

and hence $\|S(T-z)^{-1}\| < 1/2$ if $|\operatorname{Re} z| < h$, $|\operatorname{Im} z| > t_0$. By the assumptions on $T$ and (v), it follows that, with $h'' := \min\{h, h'\}$,

$$\big\|(T+S-z)^{-1}\big\| \le 2\,\big\|(T-z)^{-1}\big\| \le 2M, \quad |\operatorname{Re} z| \le h'',\ |\operatorname{Im} z| > t_0. \tag{2.7.20}$$

Since $\{z \in \mathbb{C} : -h'' < \operatorname{Re} z < h''\} \subset \rho(T+S)$, there is an $M' \ge 0$ so that

$$\big\|(T+S-z)^{-1}\big\| \le M', \quad |\operatorname{Re} z| \le h'',\ |\operatorname{Im} z| \le t_0.$$

Altogether, this shows that condition (ii) of Theorem 2.7.18 holds for $T+S$. Condition (iii) of Theorem 2.7.18 follows from (2.7.20) and the corresponding condition for $T$. Finally, we use that, by the second resolvent identity,

$$\frac{1}{\pi \mathrm{i}}\int'_{\mathrm{i}\mathbb{R}}(T+S-z)^{-1}\mathrm{d}z = \frac{1}{\pi \mathrm{i}}\int'_{\mathrm{i}\mathbb{R}}(T-z)^{-1}\mathrm{d}z - \frac{1}{\pi \mathrm{i}}\int'_{\mathrm{i}\mathbb{R}}(T+S-z)^{-1}S(T-z)^{-1}\mathrm{d}z.$$

By assumption (v), the last integral exists, even in the uniform operator topology. Hence assumption (iv) of Theorem 2.7.18 for $T$ implies condition (iv) of Theorem 2.7.18 for $T+S$. □

**Remark 2.7.20** The first inequality in (v) implies that $S$ is $T$-bounded with $T$-bound 0. In fact, for arbitrary $t \in \mathbb{R}$ and $x \in \mathcal{D}(T)$, we set $y := (T - \mathrm{i}t)x$ and obtain

$$\|Sx\| = \|S(T-\mathrm{i}t)^{-1}y\| \le \frac{\gamma}{1+|t|^\eta}\|(T-\mathrm{i}t)x\| \le \frac{\gamma\, t}{1+|t|^\eta}\|x\| + \frac{\gamma}{1+|t|^\eta}\|Tx\|;$$

letting $t \to \infty$, we see that the $T$-bound of $S$ is 0.
The second inequality in (v) is a consequence of the first one if there exists a $\gamma'' \ge 0$ such that

(vi) $\big\|(T-\mathrm{i}t)^{-1}\big\| \le \dfrac{\gamma''}{1+|t|}, \quad t \in \mathbb{R}.$

In the next two theorems, we apply the above results to dichotomous diagonally dominant and off-diagonally dominant block operator matrices. In both cases, the separability of the spectrum at $\infty$ is ensured by the assumptions that $C = B^*$ and $A$, $-D$ are regularly m-accretive with

$$\operatorname{Re} W(D) \le -\delta < 0 < \alpha \le \operatorname{Re} W(A).$$

The following theorem, which was first proved in [LT01, Theorem 4.1], concerns the diagonally dominant case; it generalizes [LT98, Theorem 4.3] where only the diagonal element $A$ was allowed to be unbounded.

**Theorem 2.7.21** *Let $A$ and $D$ be closed and suppose that there exist $\alpha$, $\delta > 0$ as well as $\varphi$, $\vartheta \in [0, \pi/2)$ such that*

$$W(A) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq \alpha, \ |\arg z| \leq \varphi\}, \tag{2.7.21}$$

$$W(D) \subset \{z \in \mathbb{C} : \operatorname{Re} z \leq -\delta, \ |\arg z| \geq \pi - \vartheta\}, \tag{2.7.22}$$

*and $\rho(A) \cap \rho(D) \cap \{z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha\} \neq \emptyset$. Assume further that $\mathcal{D}(A) \subset \mathcal{D}(B^*)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$ and that there exist $\gamma$, $\eta > 0$ such that*

$$\big\|B^*(A - \mathrm{i}t)^{-1}\big\| \leq \frac{\gamma}{1 + |t|^\eta}, \quad \big\|B(D - \mathrm{i}t)^{-1}\big\| \leq \frac{\gamma}{1 + |t|^\eta}, \quad t \in \mathbb{R}. \tag{2.7.23}$$

*Then the block operator matrix*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus \mathcal{D}(D),$$

*is closed in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$, and the following hold:*

i) *There exist projections $Q_\pm$ in $\mathcal{H}$ with*

$$\frac{1}{\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}}' (\mathcal{A} - z)^{-1} \mathrm{d}z = Q_+ - Q_-, \quad Q_+ + Q_- = I.$$

ii) *There exist strict contractions $K_+ \in L(\mathcal{H}_1, \mathcal{H}_2)$ and $K_- \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that $\mathcal{L}_\pm := R(Q_\pm)$ can be represented as*

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_- = \left\{ \begin{pmatrix} K_- y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\}$$

*and $\mathcal{H} = \mathcal{L}_+ \dot{+} \mathcal{L}_-$, $\mathcal{L}_\pm \cap \mathcal{D}(\mathcal{A}) = Q_\pm \mathcal{D}(\mathcal{A})$, $\mathcal{A}\big(\mathcal{L}_\pm \cap \mathcal{D}(\mathcal{A})\big) \subset \mathcal{L}_\pm$.*

iii) *The inclusions*

$$K_+\big(\mathcal{D}(A)\big) \subset \mathcal{D}(D), \quad K_-\big(\mathcal{D}(D)\big) \subset \mathcal{D}(A)$$

*hold and hence*

$$\mathcal{L}_+ \cap \mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{D}(A) \right\},$$
$$\mathcal{L}_- \cap \mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} K_- y \\ y \end{pmatrix} : y \in \mathcal{D}(D) \right\}.$$

***Proof.*** Assumption (2.7.23) implies that $B^*$ is $A$-bounded with $A$-bound 0 and $B$ is $D$-bounded with $D$-bound 0 (see Remark 2.7.20). Hence $\mathcal{A}$ is diagonally dominant of order 0 with closed diagonal entries and thus closed by Theorem 2.2.7 i).

Next we prove that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 2.7.5. The assumptions (2.7.21), (2.7.22) imply, by Proposition 2.7.6, that the operator $\mathcal{A}$ is uniformly $\mathcal{J}$-accretive with respect to the self-adjoint involution $\mathcal{J} = \operatorname{diag}(I, -I)$. For the second assumption of Theorem 2.7.5, we show that the operators

$$\mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \tag{2.7.24}$$

satisfy the assumptions of Theorem 2.7.19. Since, by the assumptions, $A$ and $-D$ are regularly m-accretive, the operator $\mathcal{T}$ satisfies the assumptions (i), (ii), (iii), and (iv) of Theorem 2.7.18 *e.g.* with $h := \min\{\delta, \alpha\}/2$; in fact,

$$\frac{1}{\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}}' (\mathcal{T} - z)^{-1} \mathrm{d}z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Clearly, since $\mathcal{A}$ is diagonally dominant, we have $\mathcal{D}(\mathcal{T}) \subset \mathcal{D}(\mathcal{S})$. The relation

$$\mathcal{S}(\mathcal{T} - \mathrm{i}t)^{-1} = \begin{pmatrix} 0 & B(D - \mathrm{i}t)^{-1} \\ B^*(A - \mathrm{i}t)^{-1} & 0 \end{pmatrix}$$

and (2.7.23) imply that $\mathcal{T}$ and $\mathcal{S}$ satisfy the the first growth condition in (v) of Theorem 2.7.19. Since $A$ and $-D$ are regularly m-accretive, the resolvents of $A$ and $D$ both satisfy condition (vi) in Remark 2.7.20; this follows *e.g.* from the estimate $\|(A - z)^{-1}\| \le 1/\operatorname{dist}\big(z, W(A)\big)$ for $z \notin \overline{W(A)}$. By Remark 2.7.20, also the second growth condition in (v) of Theorem 2.7.19 holds for $\mathcal{T}$. Finally, since $\mathcal{A}$ is diagonally dominant of order 0, Theorem 2.5.18 yields that $\{z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha\} \subset \rho(\mathcal{A})$; hence also the last assumption of Theorem 2.7.19 is satisfied for $\mathcal{T}$ and $\mathcal{S}$.

Now Theorem 2.7.19 and Theorem 2.7.5 apply to $\mathcal{A} = \mathcal{T} + \mathcal{S}$ and yield all claims up to ii).

It remains to prove iii). By the first and the last claim in ii), we have

$$\mathcal{L}_+ \cap \mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{D}(A),\ K_+ x \in \mathcal{D}(D) \right\}$$

and $(\mathcal{A} - z)^{-1} \mathcal{L}_+ = \mathcal{L}_+ \cap \mathcal{D}(\mathcal{A})$ for $z \in \rho(\mathcal{A})$. Therefore, if $P_1 : \mathcal{H} \to \mathcal{H}_1$ denotes the projection of $\mathcal{H}$ onto the first component, it is sufficient to prove that $P_1(\mathcal{A} - z)^{-1} \mathcal{L}_+ = \mathcal{D}(A)$ for some $z \in \rho(\mathcal{A})$. Since $\mathcal{A}$ is closed, Theorem 2.3.3 ii) shows that

$$P_1(\mathcal{A} - z)^{-1} \mathcal{L}_+ = \big\{ S_1(z)^{-1} \big(I - B(D - z)^{-1} K_+\big) x : x \in \mathcal{H}_1 \big\}.$$

By (2.7.23), there is a $t_0 \ge 0$ so that $\big\|B(D - \mathrm{i}t)^{-1}\big\| < 1$ for $|t| \ge t_0$; then $I - B(D - \mathrm{i}t)^{-1} K_+$ is a bijection on $\mathcal{H}_1$. Since $\mathrm{i}\mathbb{R} \in \rho(\mathcal{A}) \cap \rho(D) = \rho(S_1)$, the operator $S_1(\mathrm{i}t)^{-1} = \big(A - z - B(D - z)^{-1} B^*\big)^{-1}$ is a bijection from $\mathcal{H}_1$ onto $\mathcal{D}\big(S_1(\mathrm{i}t)\big) = \mathcal{D}(A)$ and hence $P_1(\mathcal{A} - \mathrm{i}t)^{-1} \mathcal{L}_+ = \mathcal{D}(A)$. □

**Remark 2.7.22** For bounded $B$, all assumptions of Theorem 2.7.21 involving $B$ are satisfied. Thus Theorem 2.7.21 generalizes and improves [LT98, Theorem 4.3], where $B$ and $D$ were supposed to be bounded; there it was only proved that the products $K_+K_-$ and $K_-K_+$ are strict contractions, while Theorem 2.7.21 shows that $K_+$ and $K_-$ are strict contractions.

As in the self-adjoint case (see Corollary 2.7.8), the invariance of the subspaces $\mathcal{L}_\pm$ together with their angular operator representations implies that the angular operators $K_\pm$ satisfy Riccati equations. Here, since the block operator matrix $\mathcal{A}$ is closed and $K_+\big(\mathcal{D}(A)\big) \subset \mathcal{D}(D)$, $K_-\big(\mathcal{D}(D)\big) \subset \mathcal{D}(A)$, the Riccati equations can be written in their standard forms and hold on the whole first and second component of the domain of $\mathcal{A}$.

**Corollary 2.7.23** *Let the diagonally dominant block operator matrix $\mathcal{A}$ satisfy the assumptions of Theorem* 2.7.21*. Then the angular operators $K_+$ and $K_-$ are solutions of the Riccati equations*

$$\begin{aligned} K_+BK_+ + K_+A - DK_+ - B^* = 0 \quad &\textit{on} \;\; \mathcal{D}(A), \\ K_-B^*K_- + K_-D - AK_- - B = 0 \quad &\textit{on} \;\; \mathcal{D}(D). \end{aligned}$$

If $\mathcal{A}$ is self-adjoint, then $\mathcal{L}_+ \perp \mathcal{L}_-$ implies $K_- = -K_+^*$ in Theorem 2.7.21 and Corollary 2.7.23; so all statements therein agree with Theorem 2.7.7 and Corollary 2.7.8, respectively. Note that, since $\mathcal{A}$ is diagonally dominant and closed, case i) or iii) of Corollary 2.7.8 prevails and $\mathcal{D}_+ = \mathcal{D}(A)$, $\mathcal{D}_- = \mathcal{D}(D)$.

Finally, we prove a theorem analogous to Theorem 2.7.21 for off-diagonally dominant block operator matrices.

**Theorem 2.7.24** *Let $B$ be boundedly invertible with $\mathcal{D}(B^*) \subset \mathcal{D}(A)$ and $\mathcal{D}(B) \subset \mathcal{D}(D)$. Suppose that $A$ and $D$ satisfy assumptions* (2.7.21) *and* (2.7.22) *and that there exist $\gamma, \eta > 0$ such that*

$$\begin{aligned} \big\|A(BB^* + t^2)^{-1}\big\| \le \frac{\gamma}{1+|t|^{\eta+1}}, \quad \big\|AB(B^*B + t^2)^{-1}\big\| \le \frac{\gamma}{1+|t|^{\eta}}, \\ \big\|D(B^*B + t^2)^{-1}\big\| \le \frac{\gamma}{1+|t|^{\eta+1}}, \quad \big\|DB^*(BB^* + t^2)^{-1}\big\| \le \frac{\gamma}{1+|t|^{\eta}}, \end{aligned} \tag{2.7.25}$$

*for $t \in \mathbb{R}$. Then the block operator matrix*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(B^*) \oplus \mathcal{D}(B),$$

*is closed in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$, and the following hold:*

i) *There exist projections $Q_\pm$ in $\mathcal{H}$ with*

$$\frac{1}{\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}}^{\prime} (\mathcal{A} - z)^{-1} \mathrm{d}z = Q_+ - Q_-, \quad Q_+ + Q_- = I.$$

ii) *There exist strict contractions* $K_+ \in L(\mathcal{H}_1, \mathcal{H}_2)$ *and* $K_- \in L(\mathcal{H}_2, \mathcal{H}_1)$ *such that* $\mathcal{L}_\pm := R(Q_\pm)$ *can be represented as*

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_- = \left\{ \begin{pmatrix} K_- y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\}$$

*and* $\mathcal{H} = \mathcal{L}_+ \dot{+} \mathcal{L}_-$, $\mathcal{L}_\pm \cap \mathcal{D}(\mathcal{A}) = Q_\pm \mathcal{D}(\mathcal{A})$, $\mathcal{A}\big(\mathcal{L}_\pm \cap \mathcal{D}(\mathcal{A})\big) \subset \mathcal{L}_\pm$.

***Proof.*** The proof of this theorem follows the lines of the proof of Theorem 2.7.21, with the roles of $\mathcal{T}$ and $\mathcal{S}$ as in (2.7.24) exchanged. In order to see that the assumptions of Theorem 2.7.18 are satisfied for $\mathcal{S}$ and those of Theorem 2.7.19 are satisfied with $\mathcal{T}$ and $\mathcal{S}$ interchanged, we observe that $(-\infty, 0] \subset \rho(B^*B)$ and hence $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{S})$,

$$(\mathcal{S} - \mathrm{i}t)^{-1} = \begin{pmatrix} -\mathrm{i}t & B \\ B^* & -\mathrm{i}t \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{i}t\,(BB^* + t^2)^{-1} & B(B^*B + t^2)^{-1} \\ B^*(BB^* + t^2)^{-1} & \mathrm{i}t\,(B^*B + t^2)^{-1} \end{pmatrix}$$

by (2.6.4) and thus

$$\mathcal{T}(\mathcal{S} - \mathrm{i}t)^{-1} = \begin{pmatrix} \mathrm{i}t\,A(BB^* + t^2)^{-1} & AB(B^*B + t^2)^{-1} \\ DB^*(BB^* + t^2)^{-1} & \mathrm{i}t\,D(B^*B + t^2)^{-1} \end{pmatrix}.$$

The growth conditions (2.7.25) imply that $\mathcal{T}$ is $\mathcal{S}$-bounded with $\mathcal{S}$-bound 0 and so $\mathcal{A}$ is off-diagonally dominant of order 0 (see Remark 2.7.20). □

In the off-diagonally dominant case, it is not clear whether an analogue of Theorem 2.7.21 iii) holds; only in special cases, including abstract Dirac operators (see Section 3.3.1), we can prove that $K_+\big(\mathcal{D}(B^*)\big) \subset \mathcal{D}(B)$ and $K_-\big(\mathcal{D}(B)\big) \subset \mathcal{D}(B^*)$.

Therefore, in a similar way as in the essentially self-adjoint case, we can write down the Riccati equations only on certain subsets $\mathcal{D}_+ \subset \mathcal{D}(B^*)$ and $\mathcal{D}_- \subset \mathcal{D}(B)$; nevertheless, since the block operator matrix $\mathcal{A}$ is closed here, the Riccati equations hold in their standard forms (compare Corollary 2.7.23 for the diagonally dominant case).

**Corollary 2.7.25** *Let the off-diagonally dominant block operator matrix* $\mathcal{A}$ *satisfy the assumptions of Theorem* 2.7.24 *and define*

$$\mathcal{D}_+ := \big\{x \in \mathcal{D}(B^*) : K_+ x \in \mathcal{D}(B)\big\}, \;\; \mathcal{D}_- := \big\{y \in \mathcal{D}(B) : K_- y \in \mathcal{D}(B^*)\big\}. \quad (2.7.26)$$

*Then the angular operators* $K_+$ *and* $K_-$ *satisfy the Riccati equations*

$$K_+ B K_+ + K_+ A - D K_+ - B^* = 0 \quad \text{on } \mathcal{D}_+,$$
$$K_- B^* K_- + K_- D - A K_- - B = 0 \quad \text{on } \mathcal{D}_-.$$

**Remark 2.7.26** Results similar to those presented in this section were proved in [LRvdR02] by H. Langer, A.C.M. Ran, and B.A. van de Rotten and, later, in [RvdM04], [vdMR05] by A.C.M. Ran and C. van der Mee:

a) In [LRvdR02], Theorem 2.7.18 was applied to another class of dichotomous block operator matrices, so-called *Hamiltonians*

$$\mathcal{A} = \begin{pmatrix} A & -D \\ -Q & -A^* \end{pmatrix}$$

where $A$ is a regularly m-accretive operator with $\operatorname{Re} W(A) \geq \alpha > 0$ and $D$, $Q$ are bounded nonnegative operators. It was shown that the corresponding Riccati equations have a bounded positive solution $\Pi_-$ and an unbounded negative solution $\Pi_+$, which are the angular operators of the spectral subspaces $\mathcal{L}_\pm$ of $\mathcal{A}$ corresponding to the left and right half plane. Here Theorem 2.7.5 has to be applied with different self-adjoint involutions. In fact, with respect to

$$\mathcal{J}_1 := \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \mathcal{J}_2 := \begin{pmatrix} 0 & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix},$$

$\mathcal{A}$ is $\mathcal{J}_1$-accretive and $\mathrm{i}\mathcal{A}$ is $\mathcal{J}_2$-self-adjoint. The self-adjointness of $\Pi_\pm$ follows from the fact that $\mathcal{L}_\pm$ are $\mathcal{J}_2$-neutral; the positivity of $\Pi_-$ and the negativity of $\Pi_+$ follow from the facts that $\mathcal{L}_-$ is $\mathcal{J}_1$-nonpositive and $\mathcal{L}_+$ is $\mathcal{J}_1$-nonnegative.

b) In [RvdM04], [vdMR05], perturbation results for exponentially dichotomous operators in Banach spaces were proved. Such operators can be written as block diagonal operator matrices $\operatorname{diag}(A_0, -A_1)$ where $A_0$, $A_1$ are generators of uniformly exponentially stable $C_0$-semigroups (see [EN00, Section V.1 b]); in this case, $\operatorname{diag}(A_0, -A_1)$ is the generator of a so-called bi-semigroup. These perturbation results yield a series of equivalent conditions for the existence of bounded solutions of the corresponding Riccati equations; they apply to block operator matrices

$$\mathcal{A} = \begin{pmatrix} A_0 & -D \\ -Q & -A_1 \end{pmatrix} = \operatorname{diag}(A_0, -A_1) + \begin{pmatrix} 0 & -D \\ -Q & 0 \end{pmatrix}$$

with bounded off-diagonal entries $D$ and $Q$. In [BvdMR05], finite dimensional approximations of the solutions of the Riccati equations are derived under the assumption that $D$ is compact.

Note that the assumptions on the diagonal entries $A$, $D$ in Theorem 2.7.21 and on $A$, $-A^*$ in [LRvdR02] imply that the block diagonal operators $\operatorname{diag}(A, D)$ and $\operatorname{diag}(A, -A^*)$, respectively, generate bi-semigroups that are even holomorphic (see [Kat95, Section IX.1.6]).

## 2.8 Block diagonalization and half range completeness

By means of the angular operators $K_+$, $K_-$ established in the previous section, we are now able to transform dichotomous block operator matrices into block diagonal form. It turns out that, restricted to the spectral subspaces $\mathcal{L}_+$, $\mathcal{L}_-$ corresponding to the right and the left half plane, the block operator matrix $\mathcal{A}$ is unitarily equivalent to the operators $A + BK_+$ and $D + B^*K_-$, respectively.

Moreover, if the spectrum *e.g.* in the right half plane is discrete, we show that the first components of the corresponding eigenvectors and associated vectors of $\mathcal{A}$ form a complete system or even a Riesz basis; these properties are also referred to as *half range completeness* or *half range basisness.*

We consider the three cases of Section 2.7, always assuming $C \subset B^*$: the essentially self-adjoint case (Theorem 2.7.7 and Remark 2.7.12), the diagonally dominant case (Theorem 2.7.21), and the off-diagonally dominant case (Theorem 2.7.24).

**Theorem 2.8.1** *Assume that $\mathcal{A}$ is essentially self-adjoint with closure $\overline{\mathcal{A}}$, the entries $A, B$, and $D$ are closed and*

$$\begin{aligned} (Ax, x) &\geq 0, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B^*), \\ (Dy, y) &\leq 0, \quad y \in \mathcal{D}(B) \cap \mathcal{D}(D). \end{aligned}$$

*Let $K$ be the angular operator from Theorem* 2.7.7 *and let the sets $\mathcal{D}_+ \subset \mathcal{H}_1$, $\mathcal{D}_- \subset \mathcal{H}_2$ be defined as in* (2.7.15). *Then $\overline{\mathcal{A}}$ admits the block diagonalization*

$$\begin{pmatrix} I & -K^* \\ K & I \end{pmatrix}^{-1} \overline{\mathcal{A}} \begin{pmatrix} I & -K^* \\ K & I \end{pmatrix} = \begin{pmatrix} Z_+ & 0 \\ 0 & Z_- \end{pmatrix} \quad \text{on } \mathcal{D}_+ \oplus \mathcal{D}_- \qquad (2.8.1)$$

*with linear operators $Z_\pm$ defined on $\mathcal{D}_\pm$; moreover,*

i) *$\overline{\mathcal{A}}\big|_{\mathcal{L}_+}$ is unitarily equivalent to the operator $Z_+$ which is self-adjoint and nonnegative in the Hilbert space $\widehat{\mathcal{H}}_1 := \big(\mathcal{H}_1, \big((I + K^*K)\,\cdot, \cdot\big)\big)$,*

ii) *$\overline{\mathcal{A}}\big|_{\mathcal{L}_-}$ is unitarily equivalent to the operator $Z_-$ which is self-adjoint and nonpositive in the Hilbert space $\widehat{\mathcal{H}}_2 := \big(\mathcal{H}_2, \big((I + KK^*)\,\cdot, \cdot\big)\big)$.*

***Proof.*** The spectral subspace $\mathcal{L}_+ = \mathcal{L}_{[0,\infty)}\big(\overline{\mathcal{A}}\big)$ is invariant under $\overline{\mathcal{A}}$, that is, $\overline{\mathcal{A}}\big(\mathcal{L}_+ \cap \mathcal{D}(\overline{\mathcal{A}})\big) \subset \mathcal{L}_+$. On the other hand, by Theorem 2.7.7 i), $\mathcal{L}_+$ admits the angular operator representation (2.7.7). Hence for every $x \in \mathcal{D}_+$, there exists a (unique) $u =: Z_+x \in \mathcal{H}_1$ such that

$$\overline{\mathcal{A}}\begin{pmatrix} I \\ K \end{pmatrix} x = \overline{\mathcal{A}}\begin{pmatrix} x \\ Kx \end{pmatrix} = \begin{pmatrix} u \\ Ku \end{pmatrix} = \begin{pmatrix} Z_+x \\ KZ_+x \end{pmatrix} = \begin{pmatrix} I \\ K \end{pmatrix} Z_+x.$$

Similarly, for every $y \in \mathcal{D}_-$ there exists a (unique) $v =: Z_- y \in \mathcal{H}_2$ such that

$$\overline{\mathcal{A}} \begin{pmatrix} -K^* \\ I \end{pmatrix} y = \begin{pmatrix} -K^* \\ I \end{pmatrix} Z_- y.$$

Since $I + K^*K$ and $I + KK^*$ are bijective, the inverse

$$\begin{pmatrix} I & -K^* \\ K & I \end{pmatrix}^{-1} = \begin{pmatrix} I & K^* \\ -K & I \end{pmatrix} \begin{pmatrix} (I + K^*K)^{-1} & 0 \\ 0 & (I + KK^*)^{-1} \end{pmatrix} \qquad (2.8.2)$$

exists and (2.8.1) follows. With $\widehat{\mathcal{H}}_1$, $\widehat{\mathcal{H}}_2$ defined as above, the operators

$$\mathcal{U}_+ : \widehat{\mathcal{H}}_1 \to \mathcal{L}_+, \;\; \mathcal{U}_+ x := \begin{pmatrix} I \\ K \end{pmatrix} x, \quad \mathcal{U}_- : \widehat{\mathcal{H}}_2 \to \mathcal{L}_-, \;\; \mathcal{U}_- y := \begin{pmatrix} -K^* \\ I \end{pmatrix} y,$$

are unitary, $\mathcal{D}_\pm = \mathcal{U}_\pm^{-1}\big(\mathcal{L}_\pm \cap \mathcal{D}(\overline{\mathcal{A}})\big)$, (2.8.1) can be written as

$$\overline{\mathcal{A}}\,(\mathcal{U}_+\;\mathcal{U}_-) = (\mathcal{U}_+\;\mathcal{U}_-) \begin{pmatrix} Z_+ & 0 \\ 0 & Z_- \end{pmatrix},$$

and hence

$$Z_\pm = \mathcal{U}_\pm^{-1} \overline{\mathcal{A}} \mathcal{U}_\pm = \mathcal{U}_\pm^{-1} \overline{\mathcal{A}}\big|_{\mathcal{L}_\pm} \mathcal{U}_\pm, \quad \mathcal{D}(Z_\pm) = \mathcal{U}_\pm^{-1}\big(\mathcal{L}_\pm \cap \mathcal{D}(\overline{\mathcal{A}})\big) = \mathcal{D}_\pm.$$

Now the claimed properties of $Z_\pm$ follow immediately from the corresponding properties of $\overline{\mathcal{A}}\big|_{\mathcal{L}_\pm}$. □

If an explicit description of the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ is available, the operators $Z_+$, $Z_-$ in Theorem 2.8.1 can be calculated. The following corollary is a direct consequence of Theorems 2.2.14, 2.2.23, 2.2.18, and 2.2.25.

**Corollary 2.8.2** *Let the essentially self-adjoint block operator matrix $\mathcal{A}$ satisfy the assumptions of Theorem* 2.8.1*. Then the operators $Z_+$, $Z_-$ in the block diagonalization of $\overline{\mathcal{A}}$ have the following forms:*

i) *if $\mathcal{D}(A) \subset \mathcal{D}(B^*)$, $\rho(A) \neq \emptyset$, and for some (and hence for all) $\mu \in \rho(A)$ the operator $(A - \mu)^{-1}B$ is bounded on $\mathcal{D}(B)$, then*

$$Z_+ = (A - \mu)\big(I + \overline{(A - \mu)^{-1}B}K\big) + \mu,$$
$$Z_- = \overline{S_2(\mu)} + \mu + B^*\big(-K^* + \overline{(A - \mu)^{-1}B}\big);$$

ii) *if $\mathcal{D}(B^*) \subset \mathcal{D}(A)$, $B^*$ is boundedly invertible, and for some (and hence for all) $\mu \in \mathbb{C}$ the operator $B^{-*}(D - \mu)$ is bounded on $\mathcal{D}(D)$, then*

$$Z_+ = (A - \mu)\big(I + \overline{B^{-*}(D - \mu)}\big) + \overline{T_2(\mu)}K + \mu,$$
$$Z_- = B^*\big(\overline{B^{-*}(D - \mu)} - K^*\big) + \mu;$$

iii) *if* $\mathcal{D}(D) \subset \mathcal{D}(B)$, $\rho(D) \neq \emptyset$, *and for some* (*and hence for all*) $\mu \in \rho(D)$ *the operator* $(D-\mu)^{-1}B^*$ *is bounded on* $\mathcal{D}(B^*)$, *then*

$$Z_+ = \overline{S_1(\mu)} + \mu + B\big(K + \overline{(D-\mu)^{-1}B^*}\big),$$
$$Z_- = (D-\mu)\big(I - \overline{(D-\mu)^{-1}B^*}K^*\big);$$

iv) *if* $\mathcal{D}(B) \subset \mathcal{D}(D)$, $B$ *is boundedly invertible, and for some* (*and hence for all*) $\mu \in \mathbb{C}$ *the operator* $B^{-1}(A-\mu)$ *is bounded on* $\mathcal{D}(A)$, *then*

$$Z_+ = B\big(\overline{B^{-1}(A-\mu)} + K\big) + \mu,$$
$$Z_- = (D-\mu)\big(I - \overline{B^{-1}(A-\mu)}K^*\big) - \overline{T_1(\mu)}K^* + \mu.$$

**Remark 2.8.3** The restrictions of $Z_\pm$ to the sets $\mathcal{D}_{0,\pm} \subset \mathcal{D}_\pm$ defined in or analogously to Remark 2.7.9 simplify to $Z_+|_{\mathcal{D}_{0,+}} = A + BK$, $Z_-|_{\mathcal{D}_{0,-}} = D - B^*K^*$; recall that $\mathcal{D}_{0,\pm} = \mathcal{D}_\pm$ if $\mathcal{A}$ is self-adjoint (and hence closed).

**Remark 2.8.4** Analogues of Theorem 2.8.1 and Corollary 2.8.2 hold for essentially $\mathcal{J}$-self-adjoint block operator matrices under the assumptions and modifications stated in Remark 2.7.17 (see [AL95], [MS96]); in particular, $B^*$ has to be replaced by $-B^*$ and $-K^*$ by $K^*$.

**Theorem 2.8.5** *Let* $\mathcal{A}$ *be a closed block operator matrix for which one of the following holds:*

(i) $\mathcal{A}$ *is diagonally dominant and fulfils the assumptions of Theorem* 2.7.21*; in this case let* $\mathcal{D}_+ = \mathcal{D}(A)$, $\mathcal{D}_- = \mathcal{D}(D)$.

(ii) $\mathcal{A}$ *is off-diagonally dominant and fulfils the assumptions of Theorem* 2.7.24*; in this case let* $\mathcal{D}_\pm$ *be defined as in* (2.7.26).

*Let the uniform contractions* $K_+$, $K_-$ *be the angular operators in the representations of the spectral subspaces* $\mathcal{L}_\pm$ *corresponding to* $\sigma_\pm := \sigma(\mathcal{A}) \cap \mathbb{C}_\pm$. *Then* $\mathcal{A}$ *admits the block diagonalization*

$$\begin{pmatrix} I & K_- \\ K_+ & I \end{pmatrix}^{-1} \mathcal{A} \begin{pmatrix} I & K_- \\ K_+ & I \end{pmatrix} = \begin{pmatrix} A+BK_+ & 0 \\ 0 & D+B^*K_- \end{pmatrix} \ \text{on} \ \mathcal{D}_+ \oplus \mathcal{D}_-;$$

*moreover, with* $\mathcal{J} := \operatorname{diag}(I, -I)$,

i) $\mathcal{A}|_{\mathcal{L}_+}$ *in the Hilbert space* $\widetilde{\mathcal{L}}_+ := \big(\mathcal{L}_+, (\mathcal{J}\cdot,\cdot)\big)$ *is unitarily equivalent to the operator* $Z_+ = A + BK_+$ *which is uniformly accretive in the Hilbert space* $\widetilde{\mathcal{H}}_1 := \big(\mathcal{H}_1, \big((I - K_+^*K_+)\cdot,\cdot\big)\big)$ *with domain* $\mathcal{D}_+$,

ii) $\mathcal{A}|_{\mathcal{L}_-}$ *in the Hilbert space* $\widetilde{\mathcal{L}}_- := \big(\mathcal{L}_-, -(\mathcal{J}\cdot,\cdot)\big)$ *is unitarily equivalent to the operator* $Z_- = D + B^*K_-$ *for which* $-Z_-$ *is uniformly accretive in the Hilbert space* $\widetilde{\mathcal{H}}_2 := \big(\mathcal{H}_2, \big((I - K_-K_-^*)\cdot,\cdot\big)\big)$ *with domain* $\mathcal{D}_-$.

***Proof.*** The proof is similar to that of Theorem 2.8.1 with the following differences: The spaces $\widetilde{\mathcal{H}}_1$, $\widetilde{\mathcal{H}}_2$, $\widetilde{\mathcal{L}}_+$, $\widetilde{\mathcal{L}}_-$ are Hilbert spaces since $K_\pm$ are uniform contractions (so that $I - K_\pm^* K_\pm$ are bijective) and since the inner product $(\mathcal{J}\cdot,\cdot)$ is positive definite on $\widetilde{\mathcal{L}}_+$ and negative definite on $\widetilde{\mathcal{L}}_-$. The operators

$$\mathcal{U}_+ : \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{L}}_+, \;\; \mathcal{U}_+ x := \begin{pmatrix} I \\ K_+ \end{pmatrix} x, \quad \mathcal{U}_- : \widetilde{\mathcal{H}}_2 \to \widetilde{\mathcal{L}}_-, \;\; \mathcal{U}_- y := \begin{pmatrix} K_- \\ I \end{pmatrix} y,$$

are unitary. To prove the claimed properties *e.g.* of $Z_+ = \mathcal{U}_+^{-1} \mathcal{A}|_{\mathcal{L}_+} \mathcal{U}_+$, we use the Riccati equation for $K_+$ on $\mathcal{D}_+$ to obtain that, for $x \in \mathcal{D}_+$,

$$\begin{aligned}\big((I - K_+^* K_+)(A + BK_+)x, x\big) &= \big((A+BK_+)x, x\big) - \big((B^* + DK_+)x, K_+ x\big) \\ &= \left(\mathcal{J}\mathcal{A}\begin{pmatrix} x \\ K_+ x\end{pmatrix}, \begin{pmatrix} x \\ K_+ x\end{pmatrix}\right);\end{aligned}$$

since $\mathcal{A}$ is uniformly $\mathcal{J}$-accretive by Proposition 2.7.6, it follows that

$$\mathrm{Re}\,\big((I - K_+^* K_+)(A + BK_+)x, x\big) \geq \beta\, \|x\|^2$$

with some constant $\beta > 0$, and similary for $\mathcal{A}|_{\mathcal{L}_-}$. □

**Remark 2.8.6** Note that, since $\mathcal{A}$ is not self-adjoint, we cannot conclude *e.g.* from $\sigma(\mathcal{A}|_{\mathcal{L}_+}) \subset \mathbb{C}_+$ that $\mathcal{A}|_{\mathcal{L}_+}$ is accretive with respect to the original scalar product $(\cdot,\cdot)$ of $\mathcal{H}$ on $\mathcal{L}_+$; this holds only with respect to the indefinite inner product $(\mathcal{J}\cdot,\cdot)$ on $\mathcal{H}$. Therefore the new scalar products on $\mathcal{H}_1$ and on $\mathcal{H}_2$ have different signs in Theorem 2.8.1 and Theorem 2.8.5.

For the rest of this section, we consider the case that *e.g.* the spectrum in the open right half plane is discrete, *i.e.* consists of countably many eigenvalues of finite algebraic multiplicities accumulating at most at $\infty$. As a consequence of the block diagonalization established in the previous theorems, we obtain a half range basisness result in the essentially self-adjoint case and a half range completeness result in the non-self-adjoint case; for the latter, we restrict ourselves to the case of bounded $B$ and $D$.

In the following, a system $(z_i)_{i=1}^N \subset \mathcal{H}$, $N \in \mathbb{N} \cup \{\infty\}$, is called a *Riesz basis* of a Hilbert space $\mathcal{H}$ if it is equivalent to an orthonormal basis of $\mathcal{H}$, *i.e.* if there exists a bounded and boundedly invertible linear operator $\mathcal{G}$ in $\mathcal{H}$ and an orthonormal basis $(e_i)_{i=1}^N$ of $\mathcal{H}$ such that $z_i = \mathcal{G} e_i$, $i = 1, \dots, N$ (see [GK69, Chapter VI], also for other equivalent definitions).

**Theorem 2.8.7** *Assume that $\mathcal{A}$ is essentially self-adjoint with closure $\overline{\mathcal{A}}$, the entries $A$, $D$ are self-adjoint, $B$ is closed such that $\mathcal{D}(B) \cap \mathcal{D}(D)$ is a core of $D$, there exists $\delta > 0$ such that*

$$
\begin{aligned}
(Ax, x) &\geq 0, & x &\in \mathcal{D}(A), \\
(Dx, x) &\leq -\delta \, \|x\|^2, & x &\in \mathcal{D}(B) \cap \mathcal{D}(D),
\end{aligned}
$$

*$\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B^*)$, and $A$ has compact resolvent. Then $\sigma_{\mathrm{ess}}(\overline{\mathcal{A}}) \subset (-\infty, -\delta]$ and $\sigma(\overline{\mathcal{A}}) \cap [0, \infty)$ is discrete. If $(\lambda_i)_{i=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, is the sequence of eigenvalues of $\overline{\mathcal{A}}$ in $[0, \infty)$ (counted with geometric mulitplicities) and $(\mathbf{x}_i)_{i=1}^N \subset \mathcal{D}(\overline{\mathcal{A}})$, $\mathbf{x}_i = (x_i \ y_i)^{\mathrm{t}}$, is a corresponding system of eigenvectors, then the system $(x_i)_{i=1}^N$ of their first components forms a Riesz basis of $\mathcal{H}_1$.*

***Proof.*** The assumptions imply that all conditions of Corollary 2.4.13 are satisfied; in particular, for $\mu \in \mathbb{C} \setminus \mathbb{R}$, the operator $(D-\mu)^{-1}B^*(A-\mu)^{-1} = (D-\mu)^{-1}B^*(A-\mu)^{-1/2}(A-\mu)^{-1/2}$ is compact since $(D-\mu)^{-1}$ as well as $B^*(A-\mu)^{-1/2}$ are bounded and $(A-\mu)^{-1/2}$ is compact (the latter follows *e.g.* from [Kat95, Theorem V.3.49]). Hence, by Corollary 2.4.13,

$$
\begin{aligned}
\sigma_{\mathrm{ess}}(\overline{\mathcal{A}}) &= \sigma_{\mathrm{ess}}(A) \cup \sigma_{\mathrm{ess}}\big(D - \overline{B^*(A-\mu)^{-1}B}\big) \\
&= \sigma_{\mathrm{ess}}\big(D - \overline{B^*(A-\mu)^{-1/2}(A-\mu)^{-1/2}B}\big) \subset (-\infty, -\delta]
\end{aligned}
$$

as $A$ has compact resolvent, $\overline{B^*(A-\mu)^{-1/2}(A-\mu)^{-1/2}B} \geq 0$, and $D \leq -\delta$. Since $\overline{\mathcal{A}}$ is self-adjoint, $\sigma(\overline{\mathcal{A}}) \cap [0, \infty) \subset \sigma(\overline{\mathcal{A}}) \setminus \sigma_{\mathrm{ess}}(\overline{\mathcal{A}})$ is discrete.

By Theorem 2.8.1 i) and its proof, an element $\mathbf{x}_i = (x_i \ y_i)^{\mathrm{t}} \in \mathcal{D}(\overline{\mathcal{A}})$ is an eigenvector corresponding to an eigenvalue $\lambda_i \in \sigma(\overline{\mathcal{A}}) \cap [0, \infty)$ if and only if $y_i = Kx_i$ where $K$ is the angular operator from Theorem 2.7.7 and $x_i = \mathcal{U}_+^{-1}\mathbf{x}_i \in \mathcal{D}_+$ is an eigenvector of $Z_+$ in $\mathcal{H}_1$ corresponding to $\lambda_i$. Since $Z_+$ is self-adjoint in the Hilbert space $\widehat{\mathcal{H}}_1 := \big(\mathcal{H}_1, \big((I + K^*K)\,\cdot, \cdot\big)\big)$ and $\sigma(Z_+) = \sigma(\overline{\mathcal{A}}) \cap [0, \infty)$ is discrete, the system $(x_i)_{i=1}^N$ of eigenvectors of $Z_+$ is an orthonormal basis in $\widehat{\mathcal{H}}_1$. Then $\big((I+K^*K)^{1/2}x_i\big)_{i=1}^N$ is an orthonormal basis of $\mathcal{H}_1$ with its original scalar product. Because $(I + K^*K)^{1/2}$ is bounded and boundedly invertible, $(x_i)_{i=1}^N$ is a Riesz basis of $\mathcal{H}_1$. □

**Remark 2.8.8** The first half range basis result was proved in [AL95, Theorem 3.5] for the case that $B$ and $D$ are bounded; generalizations to unbounded upper dominant block operator matrices were given in [ALMS96, Theorem 6.4], and in [MS96, Corollary 2.6] for bounded $D$ under the weaker assumption that the spectra of $A$ and of the second Schur complement $S_2$ are separated (see Remark 2.7.17).

For non-self-adjoint operators, basis results are much more difficult to prove and often depend on detailed information about the asymptotic behaviour of eigenvalues, eigenfunctions, and of the resolvent. Nevertheless, there exist completeness results that require less knowledge (see [GK69, Chapter V] for a comprehensive presentation). We follow the lines of [LT98, Theorem 5.1] and apply a theorem by V.B. Lidskii for dissipative trace class operators (see [Lid59a], [Lid59b]). Here we denote by $\mathcal{S}_p$ the von Neumann–Schatten classes of compact operators (see [GK69, Chapter III]); in particular, $\mathcal{S}_1$ is the class of nuclear or trace class operators.

**Theorem 2.8.9** *Let $\mathcal{A}$ be a block operator matrix such that $A$ has compact resolvent with $(A-z)^{-1} \in \mathcal{S}_1$ for some (and hence for all) $z \in \rho(A)$, $B$, $D$ are bounded, $C = B^*$, and assume there exist $\alpha$, $\delta > 0$, $\varphi \in [0, \pi/2)$ with*

$$W(A) \subset \big\{ z \in \mathbb{C} : \ \operatorname{Re} z \geq \alpha, \ |\arg z| \leq \varphi \big\},$$
$$W(D) \subset \big\{ z \in \mathbb{C} : \ \operatorname{Re} z \leq -\delta \big\}.$$

*Then $\sigma_{\rm ess}(\mathcal{A}) = \sigma_{\rm ess}(D) \subset \big\{ z \in \mathbb{C} : \operatorname{Re} z \leq -\delta \big\}$ and $\sigma(\mathcal{A}) \cap \mathbb{C}_+ \subset \big\{ z \in \mathbb{C} : \operatorname{Re} z \geq \alpha \big\}$ is discrete. If $(\lambda_i)_{i=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, is the sequence of eigenvalues of $\mathcal{A}$ in $\mathbb{C}_+$ (counted with geometric multiplicities) and if $(\mathbf{x}_{ij})_{i=1,j=1}^{N \ \ k_i} \subset \mathcal{D}(\mathcal{A})$, $\mathbf{x}_{ij} = (x_{ij} \ y_{ij})^{\rm t}$, is a corresponding system of eigenvectors and associated vectors, then the system $(x_{ij})_{i=1,j=1}^{N \ \ k_i}$ of their first components is complete in $\mathcal{H}_1$.*

***Proof.*** By the assumptions, $\mathcal{A}$ is closed and diagonally dominant, and all conditions of Theorem 2.7.21 and hence of Theorem 2.8.5 are satisfied. By Theorem 2.4.8, we have $\sigma_{\rm ess}(\mathcal{A}) = \sigma_{\rm ess}(D)$ as $A$ has compact resolvent and $B$ is bounded; in particular, $\sigma_{\rm ess}(\mathcal{A})$ is bounded and so $\mathbb{C} \setminus \sigma_{\rm ess}(\mathcal{A})$ consists of one component. Since also $\sigma(\mathcal{A}) \cap \mathbb{C}_- = \sigma(D + B^* K_-)$ is bounded and hence $\rho(\mathcal{A}) \cap \mathbb{C}_- \neq \emptyset$, Theorem 2.1.10 implies that $\sigma(\mathcal{A}) \cap \mathbb{C}_+ \subset \sigma(\mathcal{A}) \setminus \sigma_{\rm ess}(\mathcal{A})$ is discrete. By Theorem 2.8.5 and Theorem 2.5.18, we have $\sigma(\mathcal{A}) \cap \mathbb{C}_+ = \sigma(A + BK_+) \subset \{ z \in \mathbb{C} : \operatorname{Re} z \geq \alpha \}$ and $(\mathbf{x}_{ij})_{j=1}^{k_i} \subset \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus \mathcal{H}_2$ is a Jordan chain of $\mathcal{A}$ at an eigenvalue $\lambda_i \in \sigma(\mathcal{A}) \cap \mathbb{C}_+$ if and only if $\mathbf{x}_{ij} = (x_{ij} \ K_+ x_{ij})^{\rm t}$ and $(x_{ij})_{j=1}^{k_i}$ is a Jordan chain of $A + BK_+ = Z_+$ at $\lambda_i$. This follows from the fact that, with $x_{i,-1} := 0$, the first equation in

$$\begin{pmatrix} A - \lambda_i & B \\ B^* & D - \lambda_i \end{pmatrix} \begin{pmatrix} x_{ij} \\ K_+ x_{ij} \end{pmatrix} = \begin{pmatrix} x_{i,j-1} \\ K_+ x_{i,j-1} \end{pmatrix}, \quad j = 1, \dots, k_i,$$

is the relation $\big(A + BK_+ - \lambda_i\big) x_{ij} = x_{i,j-1}$, and the second equation amounts to the Riccati equation for $K_+$ applied to $x_{ij}$ (see Corollary 2.7.23).

By assumption, $A$ is regularly m-accretive with $(A - \zeta)^{-1} \in \mathcal{S}_1$ for all $\zeta \in (-\infty, \alpha)$ and $B$, $K_+$ are bounded. Thus, for $\zeta \in \big(-\infty, \alpha - \|BK_+\|\big)$, the operator $A + BK_+ - \zeta$ is regularly m-accretive; more exactly, we have $\operatorname{Re} \big( (A + BK_+ - \zeta) x, x \big) \geq 0$ for $x \in \mathcal{D}(A)$, $\zeta \in \rho(A + BK_+) \cap \rho(A)$, and

$$(A + BK_+ - \zeta)^{-1} = (A - \zeta)^{-1} \big( I + BK_+ (A - \zeta)^{-1} \big)^{-1} \in \mathcal{S}_1.$$

So $\mathrm{i}(A+BK_+ +\zeta)^{-1}\in\mathcal{S}_1$ is dissipative, *i.e.* $\operatorname{Im}\big(\mathrm{i}(A+BK_+ -\zeta)^{-1}y,y\big)\geq 0$ for $y\in\mathcal{H}_1$. By Lidskii's theorem (see [GK69, Theorem V.2.3]), the eigenvectors and associated vectors of $\mathrm{i}(A+BK_+ +\zeta)^{-1}$ are complete in $\mathcal{H}_1$. Since $\lambda_i\in\sigma(A+BK_+)$ if and only if $\mathrm{i}/(\lambda_i-\zeta)\in\sigma\big(\mathrm{i}(A+BK_+ -\zeta)^{-1}\big)$ and the corresponding spectral subspaces coincide,

$$\mathcal{L}_{\lambda_i}(A+BK_+)=\mathcal{L}_{\mathrm{i}/(\lambda_i-\zeta)}\big(\mathrm{i}(A+BK_+ -\zeta)^{-1}\big),$$

the completeness in $\mathcal{H}_1$ of the system $(x_{ij})_{i=1,j=1}^{N\ \ k_i}$ of eigenvectors and associated vectors of $A+BK_+$ follows. □

## 2.9 Uniqueness results for solutions of Riccati equations

Solutions of Riccati equations have been derived by means of factorizations of the Schur complements in the bounded case (see Theorem 1.7.1) and by means of an invariant subspace approach in the unbounded dichotomous case (see Theorems 2.7.7, 2.7.21, 2.7.24). However, only existence, not uniqueness of these solutions has been obtained so far.

In this section we present two different methods to prove uniqueness results: The first method uses Banach's fixed point theorem and does not require any symmetry or dichotomy assumptions. The second method for self-adjoint block operator matrices relates to the invariant subspace approach in Section 2.7. Here we restrict ourselves to the case that only the diagonal entry $A$ is unbounded (see [ALT01]); in this case, the Riccati equation has the form

$$KBK+KA-DK-C=0 \quad \text{on } \mathcal{D}(A). \tag{2.9.1}$$

The fixed point method relies on rewriting the Riccati equation in the form of a so-called *Krein-Rosenblum equation* (sometimes also called Sylvester equation)

$$KA-DK=Y, \qquad Y:=C-KBK, \tag{2.9.2}$$

or

$$K(A+BK)-DK=Y, \quad Y:=C. \tag{2.9.3}$$

Solutions $K$ to such operator equations in integral form seem to have been found first by M.G. Krein in 1948 (see [Phó91]) and later, independently, by Yu. Daleckii (see [Dal53]) and M. Rosenblum (see [Ros56]). The crucial condition here is that the spectra of the operator coefficients on the left hand side (*e.g.* $A$ and $D$ in (2.9.2)) have to be disjoint. In our case, either one of these coefficients or the right hand side $Y$ contains the solution $K$; this yields integral equations for $K$ which we will use for fixed point theorems.

The first to apply the fixed point approach to obtain solutions of Riccati equations associated with self-adjoint block operator matrices (and, as a consequence, factorizations of the Schur complements and block diagonalizability) was A.K. Motovilov in [Mot91], [Mot95]. He studied the case of unbounded $A$ and $D$ and bounded coupling $B$, motivated by two-channel Hamiltonians from elementary particle physics (see *e.g.* [Sch87]).

In the following we consider integrals of an operator valued function $F$ with respect to a spectral function $E$. If $F$ satisfies a Lipschitz condition on $[a,b]$, these integrals can be defined as the limits of the corresponding Riemann–Stieltjes sums in the strong operator topology (see [AG93, Section 7], [ALMS96, Section 7] or [AMM03]). In particular, if the function $F$ is continuously differentiable, the integration by parts formula holds.

The next theorem uses the form (2.9.2) of the Riccati equation (2.9.1).

**Proposition 2.9.1** *Suppose that $\rho(A) \neq \emptyset$, $B$, $C$, $D$ are bounded, and*

$$\sigma(A) \cap \sigma(D) = \emptyset. \tag{2.9.4}$$

*Let $\Gamma_D$ be a Cauchy contour around $\sigma(D)$ separating it from $\sigma(A)$. Then $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ is a solution of the Riccati equation* (2.9.1) *if and only if*

$$K = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_D} (D-z)^{-1}(C - KBK)(A-z)^{-1}\,\mathrm{d}z =: \Phi_R(K); \tag{2.9.5}$$

*if $D = D^*$ with spectral function $E_D$, an equivalent condition is*

$$K = \int_{\sigma(D)} E_D(\mathrm{d}\mu)\,(C - KBK)(A-\mu)^{-1} =: \Psi_R(K). \tag{2.9.6}$$

***Proof.*** The equivalence of (2.9.1) and (2.9.5) is well-known (compare [DK74a, Theorem I.3.2] or [GGK90, Theorem I.4.1]); we prove it here for the convenience of the reader. If $K$ solves (2.9.1), then, for every $z \in \mathbb{C}$, $K(A-z) - (D-z)K = C - KBK$. Multiplying by $(A-z)^{-1}$ from the right and by $(D-z)^{-1}$ from the left, we find that, for $z \in \rho(A) \cap \rho(D)$,

$$(D-z)^{-1}K - K(A-z)^{-1} = (D-z)^{-1}(C-KBK)(A-z)^{-1}.$$

If we integrate along $\Gamma_D$, multiply by $-1/(2\pi\mathrm{i})$, and observe (2.9.4), we arrive at (2.9.5). Vice versa, it is easy to check that the expression on the right hand side of (2.9.5) satisfies the Riccati equation (2.9.1).

In the special case $D = D^*$, we rewrite the right hand side of (2.9.5) as

$$\begin{aligned} &-\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_D} \int_{\sigma(D)} (\mu - z)^{-1} E_D(\mathrm{d}\mu)\,(C-KBK)(A-z)^{-1}\,\mathrm{d}z \\ &= \int_{\sigma(D)} E_D(\mathrm{d}\mu)\,(C-KBK)\left(-\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_D} (\mu-z)^{-1}(A-z)^{-1}\,\mathrm{d}z\right) \end{aligned}$$

$$= \int_{\sigma(D)} E_D(\mathrm{d}\mu)\,(C - KBK)(A - \mu)^{-1},$$

which proves the equivalence of (2.9.5) and (2.9.6). □

The following corollary is immediate from the integral equation (2.9.5).

**Corollary 2.9.2** *If, under the assumptions of Proposition* 2.9.1, *either* $B$ *and* $C$ *are compact* ($B, C \in \mathcal{S}_p$ *for some* $p \in [1, \infty]$) *or* $A$ *has compact resolvent* ($(A-z)^{-1} \in \mathcal{S}_p$ *for some* $p \in [1, \infty]$), *then* $K$ *is compact* ($K \in \mathcal{S}_p$).

A different integral equation is obtained in the next theorem which uses the equivalent form (2.9.3) of the Riccati equation (2.9.1); the proof is completely analogous to the proof of Theorem 2.9.1.

**Proposition 2.9.3** *Suppose that* $\rho(A) \neq \emptyset$, $B$, $C$, $D$ *are bounded, and let* $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ *be such that*

$$\sigma(A + BK) \cap \sigma(D) = \emptyset. \tag{2.9.7}$$

*Let* $\Gamma_D$ *be a Cauchy contour around* $\sigma(D)$ *separating it from* $\sigma(A + BK)$. *Then* $K$ *is a solution of the Riccati equation* (2.9.1) *if and only if*

$$K = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_D} (D - z)^{-1} C (A + BK - z)^{-1}\,\mathrm{d}z =: \Phi_M(K); \tag{2.9.8}$$

*if* $D = D^*$ *with spectral function* $E_D$, *an equivalent condition is*

$$K = \int_{\sigma(D)} E_D(\mathrm{d}\mu)\, C (A + BK - \mu)^{-1} =: \Psi_M(K). \tag{2.9.9}$$

**Corollary 2.9.4** *If, under the assumptions of Proposition* 2.9.3, *either* $C$ *is compact* ($C \in \mathcal{S}_p$ *for some* $p \in [1, \infty]$) *or* $A$ *has compact resolvent* ($(A-z)^{-1} \in \mathcal{S}_p$ *for some* $p \in [1, \infty]$), *then* $K$ *is compact* ($K \in \mathcal{S}_p$).

**Remark 2.9.5** The relation (2.9.8) coincides with formula (1.7.2) in Theorem 1.7.1 for bounded block operator matrices; the different sign is due to the opposite orientation of the contour $\Gamma_1$ therein compared to $\Gamma_D$ here.

The integral equation (2.9.5) may be viewed as a fixed point equation for $K$. To ensure that the corresponding mapping $\Phi_R$ is a contraction, we assume that the spectra of $A$ and $D$ have positive distance and $B$, $C$ satisfy certain smallness assumptions.

**Theorem 2.9.6** *Suppose that* $\rho(A) \neq \emptyset$, $B$, $C$, $D$ *are bounded, and*

$$\operatorname{dist}\big(\sigma(A), \sigma(D)\big) > 0. \tag{2.9.10}$$

*Let* $\Gamma_D$ *be a Cauchy contour of length* $l_{\Gamma_D}$ *around* $\sigma(D)$ *separating it from* $\sigma(A)$*, denote*

$$a_{\Gamma_D} := \max_{z\in\Gamma_D}\big\|(A-z)^{-1}\big\|, \quad d_{\Gamma_D} := \max_{z\in\Gamma_D}\big\|(D-z)^{-1}\big\|, \tag{2.9.11}$$

*and assume that*

$$\frac{1}{2\pi}a_{\Gamma_D}d_{\Gamma_D}l_{\Gamma_D}\big(\|B\|+\|C\|\big)<1, \quad \frac{1}{\pi}a_{\Gamma_D}d_{\Gamma_D}l_{\Gamma_D}\|B\|<1. \tag{2.9.12}$$

*Then the following hold:*

i) *There exists a unique contractive solution* $K_0 \in L(\mathcal{H}_1,\mathcal{H}_2)$ *of the Riccati equation* (2.9.1)*, which is even a uniform contraction.*

ii) *For every contraction* $K_1 \in L(\mathcal{H}_1,\mathcal{H}_2)$*, the operators*

$$K_n := \Phi_R(K_{n-1}), \quad n=2,3,\dots,$$

*are uniform contractions converging to* $K_0$ *in the operator norm.*

iii) *If the inequalities in assumption* (2.9.12) *are not strict* (*i.e. if* $<$ *is replaced by* $\le$ ) *and the operator* $B$ *is compact, then the Riccati equation* (2.9.1) *has at least one contractive solution.*

***Proof.*** i), ii) By $\mathcal{K}$ we denote the set of all contractions from $\mathcal{H}_1$ into $\mathcal{H}_2$. The first inequality in (2.9.12) implies that, for each contraction $K\in\mathcal{K}$, the image $\Phi_R(K)$ (see (2.9.5)) is a uniform contraction, *i.e.* $\|\Phi_R(K)\|<1$.

Moreover, for contractions $K$, $\widetilde{K}\in\mathcal{K}$, the relation

$$\Phi_R(K)-\Phi_R(\widetilde{K})=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_D}(D-z)^{-1}\big((K-\widetilde{K})BK+\widetilde{K}B(K-\widetilde{K})\big)(A-z)^{-1}\mathrm{d}z$$

and the second inequality in (2.9.12) imply that

$$\|\Phi_R(K)-\Phi_R(\widetilde{K})\|\le\gamma\,\|K-\widetilde{K}\| \quad \text{with} \quad \gamma:=\frac{1}{\pi}a_{\Gamma_D}d_{\Gamma_D}l_{\Gamma_D}\|B\|<1.$$

Thus $\Phi_R$ is a contraction in $\mathcal{K}$ and Banach's fixed point theorem yields i) and ii).

iii) If in (2.9.12) the signs $<$ are replaced by $\le$, we consider the operators $B_n := (1-n^{-1})\,B$, $C_n := (1-n^{-1})\,C$ for $n\in\mathbb{N}$. Then, by i), for every $n\in\mathbb{N}$ there exists a (unique and uniform) contraction $K_0^{(n)}$ such that

$$K_0^{(n)}B_nK_0^{(n)}x+K_0^{(n)}Ax-DK_0^{(n)}x-C_nx=0, \quad x\in\mathcal{D}(A). \tag{2.9.13}$$

Because the unit ball in $L(\mathcal{H}_1,\mathcal{H}_2)$ is weakly compact, the sequence $\big(K_0^{(n)}\big)_1^\infty$ contains a subsequence which converges in the weak operator topology of $\mathcal{K}$ to some contraction $K_0$. Since $B$ is compact, the corresponding subsequence of $B_nK_0^{(n)}$ converges strongly to $BK_0$ and that of

$K_0^{(n)} B_n K_0^{(n)}$ converges weakly to $K_0 B K_0$. Altogether, we see that $K_0$ is a solution of the Riccati equation (2.9.1). □

Under different smallness assumptions on $B$ and $C$, also the mapping $\Phi_M$ (see (2.9.8)) can be used to prove the existence and uniqueness of contractive solutions of the Riccati equation (2.9.1).

**Theorem 2.9.7** *Suppose that $\rho(A) \neq \emptyset$, $B, C, D$ are bounded, and* $\operatorname{dist}\big(\sigma(A), \sigma(D)\big) > 0$. *Let $\Gamma_D$ be a Cauchy contour of length $l_{\Gamma_D}$ around $\sigma(D)$ separating it from $\sigma(A)$ and let $a_{\Gamma_D}$ and $d_{\Gamma_D}$ be defined as in* (2.9.11). *If $a_{\Gamma_D}\|B\| < 1$ and*

$$\frac{a_{\Gamma_D} d_{\Gamma_D} l_{\Gamma_D}}{2\pi} \frac{\|C\|}{1 - a_{\Gamma_D}\|B\|} < 1, \quad \frac{a_{\Gamma_D}^2 d_{\Gamma_D} l_{\Gamma_D}}{2\pi} \frac{\|B\|\,\|C\|}{(1 - a_{\Gamma_D}\|B\|)^2} < 1, \tag{2.9.14}$$

*then Theorem* 2.9.6 *continues to hold with $\Phi_M$ instead of $\Phi_R$.*

***Proof.*** The proof is analogous to the proof of Theorem 2.9.6 if we observe that the inequality $a_{\Gamma_D}\|B\| < 1$ implies that, for arbitrary $z \in \sigma(D)$, the resolvent $(A+BK-z)^{-1}$ exists and thus condition (2.9.7) holds; in fact,

$$\big\|(A + BK - z)^{-1}\big\| = \big\|(A - z)^{-1}(BK(A - z)^{-1} + I)^{-1}\big\| \leq \frac{a_{\Gamma_D}}{1 - a_{\Gamma_D}\|B\|}.$$

The first assumption in (2.9.14) ensures that $\Phi_M$ maps $\mathcal{K}$ into itself, the second condition in (2.9.14) guarantees that $\Phi_M$ is a contraction. □

If $D = D^*$, the mappings $\Psi_M$ and $\Psi_R$ (see (2.9.6), (2.9.9), respectively) may also be used for the fixed point theorem.

**Theorem 2.9.8** *Suppose that $\rho(A) \neq \emptyset$, $B, C, D$ are bounded, and* $\operatorname{dist}\big(\sigma(A), \sigma(D)\big) > 0$. *If $D = D^*$, $C \in \mathcal{S}_2$, and if*

$$\alpha_D := \max_{\mu \in \sigma(D)} \big\|(A - \mu)^{-1}\big\|$$

*and the Hilbert-Schmidt norm $\|C\|_2$ of $C$ satisfy $\alpha_D\|B\| < 1$ and*

$$\|C\|_2 \frac{\alpha_D}{1 - \alpha_D\|B\|} < 1, \quad \|B\|\,\|C\|_2 \left(\frac{\alpha_D}{1 - \alpha_D\|B\|}\right)^2 < 1, \tag{2.9.15}$$

*then Theorem* 2.9.6 *continues to hold with $\Psi_M$ instead of $\Phi_R$.*

***Proof.*** The roles of the inequality $\alpha_D\|B\| < 1$ and of the two inequalities in (2.9.15) are completely analogous to those of the corresponding inequalities in Theorem 2.9.7 and its proof. In addition, estimates for integrals of certain operator-valued functions are used; for details we refer the reader to [ALT01, Theorem 4.6] and its proof. □

**Remark 2.9.9** For the self-adjoint case $A = A^*$, $D = D^*$, and bounded $C = B^*$, Theorem 2.9.8 was first proved in [Mot95, Theorem 1, Corollary 1]; then the inequalities in Theorem 2.9.8 reduce to the single inequality $\|B\|_2 < \operatorname{dist}\bigl(\sigma(A),\, \sigma(D)\bigr)/2$. A corresponding result for arbitrary bounded $B, C$ was proved by S. Albeverio, K.A. Makarov, and A.K. Motovilov (see [AMM03, Theorem 3.7]); in addition, in the case $C = B^*$ they studied the spectral shift function, regarding the off-diagonal part as a perturbation.

**Remark 2.9.10** The following variants of Theorem 2.9.8 can be proved:

i) If $B,\, C \in \mathcal{S}_2$, then an analogue of Theorem 2.9.8 holds for $\Psi_R$ under smallness assumptions on $\|B\|_2$, $\|C\|_2$ similar to the ones in (2.9.15).
ii) If neither $B$ nor $C$ belong to $\mathcal{S}_2$, then integration by parts in the integrals defining $\Psi_R$ and $\Psi_M$, respectively, yields conditions on $\|B\|$, $\|C\|$ so that Theorem 2.9.6 continues to hold with $\Psi_R$ and $\Psi_M$, respectively (see [ALT01, Theorem 4.8]).

Uniqueness results may also be obtained from the relation of solutions of Riccati equations and invariant subspaces of block operator matrices

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus \mathcal{H}_2.$$

**Definition 2.9.11** Let $T$ be linear operator in a Hilbert space $\mathcal{H}$. We call a closed subspace $\mathcal{L} \subset \mathcal{H}$

i) *invariant subspace* of $T$ or $T$*-invariant subspace* if $T\bigl(\mathcal{D}(T) \cap \mathcal{L}\bigr) \subset \mathcal{L}$,
ii) *reducing subspace* of $T$ or $T$*-reducing subspace* if $\mathcal{L}$ and $\mathcal{L}^\perp$ are invariant subspaces of $T$ and $\mathcal{D}(T) = \bigl(\mathcal{D}(T) \cap \mathcal{L}\bigr) \oplus \bigl(\mathcal{D}(T) \cap \mathcal{L}^\perp\bigr)$.

If $\mathcal{L}$ is a $T$-reducing subspace, then $T = \operatorname{diag}\bigl(T|_{\mathcal{L}}, T|_{\mathcal{L}^\perp}\bigr)$ is block diagonal with respect to the decomposition $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^\perp$ and $\sigma(T) = \sigma\bigl(T|_{\mathcal{L}}\bigr) \cup \sigma\bigl(T|_{\mathcal{L}^\perp}\bigr)$ (see [Wei00, Section 2.5], [Wei80, Section 7.4, Exercise 5.39]).

**Proposition 2.9.12** *Suppose that $A$ is closed and $B, C, D$ are bounded. Then $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ is a solution of the Riccati equation* (2.9.1) *if and only if its graph subspace*

$$\mathcal{G}(K) := \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1 \right\}$$

*is $\mathcal{A}$-invariant.*

***Proof.*** If $\mathcal{G}(K)$ is $\mathcal{A}$-invariant, then, for every $x \in \mathcal{D}(A)$, there exists a $y \in \mathcal{H}_1$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ Kx \end{pmatrix} = \begin{pmatrix} y \\ Ky \end{pmatrix}. \tag{2.9.16}$$

The first component of this equality yields $y = (A + BK)x$; inserting this into the second component of (2.9.16) gives

$$(C + DK)x = K(A + BK)x, \quad x \in \mathcal{D}(A),$$

which is (2.9.1). Conversely, if $K$ solves (2.9.1), then for every $x \in \mathcal{D}(A)$ the relation (2.9.16) holds with $y = (A + BK)x$. □

**Remark 2.9.13** If $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ is a solution of the Riccati equation (2.9.1), then $KA$ is bounded on the dense subset $\mathcal{D}(A) \subset \mathcal{H}_1$. Thus $\overline{KA}$ and $(KA)^* = A^*K^*$ are bounded and everywhere defined. Therefore $K^* \in L(\mathcal{H}_2, \mathcal{H}_1)$ has the property $R(K^*) \subset \mathcal{D}(A^*)$, it satisfies the Riccati equation

$$K^*B^*K^* + A^*K^* - K^*D^* - C^* = 0 \quad \text{on } \mathcal{H}_2$$

related to the adjoint block operator matrix $\mathcal{A}^*$, and the subspace

$$\mathcal{G}(K)^\perp = \left\{ \begin{pmatrix} -K^*y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\} \subset \mathcal{D}(A^*) \oplus \mathcal{H}_2 = \mathcal{D}(\mathcal{A}^*)$$

is $\mathcal{A}^*$-invariant.

In the sequel we concentrate on Riccati equations associated with self-adjoint block operator matrices. In this case, contractive solutions correspond to invariant subspaces that are definite with respect to some indefinite inner product (see Definition 2.7.1 and Remark 2.7.2).

**Theorem 2.9.14** *Let $A = A^*$, $D = D^*$, let $B$, $D$ be bounded, and let $C = B^*$. Then the Riccati equation*

$$KBK + KA - DK - B^* = 0 \quad \textit{on } \mathcal{D}(A) \tag{2.9.17}$$

*has a contractive solution $K$ if and only if there exists a self-adjoint involution $\mathcal{J}$ in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that*

$$\mathcal{A}\mathcal{J} \;=\; \mathcal{J}\mathcal{A} \tag{2.9.18}$$

*and the subspace $\mathcal{H}_1 \oplus \{0\}$ is maximal $\mathcal{J}$-nonnegative. In this case, the contraction $K$ in* (2.9.17) *is strict* (*uniform, respectively*) *if and only if the subspace $\mathcal{H}_1 \oplus \{0\}$ is maximal $\mathcal{J}$-positive* (*maximal uniformly $\mathcal{J}$-positive, respectively*)*.*

***Proof.*** Let $K$ be a contractive solution of (2.9.17). If we denote by $P_+$ the orthogonal projection on $\mathcal{G}(K)$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and set $P_- := I - P_+$, then $\mathcal{J} := P_+ - P_-$ is a self-adjoint involution. In fact, it is easy to see that

$$
\begin{aligned}
P_+ &= \begin{pmatrix} (I+K^*K)^{-1} & (I+K^*K)^{-1}K^* \\ K(I+K^*K)^{-1} & K(I+K^*K)^{-1}K^* \end{pmatrix}, \\
P_- &= \begin{pmatrix} K^*(I+KK^*)^{-1}K & -K^*(I+KK^*)^{-1} \\ -(I+KK^*)^{-1}K & (I+KK^*)^{-1} \end{pmatrix}.
\end{aligned}
\tag{2.9.19}
$$

By Remark 2.9.13, we have $R(K^*) \subset \mathcal{D}(A)$ and hence $(I+K^*K)\,\mathcal{D}(A) = \mathcal{D}(A)$. Together with (2.9.19), we obtain $P_+\mathcal{D}(\mathcal{A}) = P_+\big(\mathcal{D}(A) \oplus \mathcal{H}_2\big) = \mathcal{D}(A) \oplus \mathcal{H}_2$ and thus $\mathcal{D}\big(P_+\mathcal{A}\big) = \mathcal{D}(\mathcal{A}) = \mathcal{D}\big(\mathcal{A}P_+\big)$. By Proposition 2.9.12 and Remark 2.9.13, $\mathcal{G}(K) = R(P_+)$ and $\mathcal{G}(K)^\perp = R(I-P_+)$ are $\mathcal{A}$-invariant so that $\mathcal{A}P_+ = P_+\mathcal{A}P_+$ and $\mathcal{A}(I-P_+) = (I-P_+)\mathcal{A}(I-P_+)$. Since $\mathcal{D}\big(P_+\mathcal{A}\big) = \mathcal{D}\big(\mathcal{A}P_+\big)$, the latter is equivalent to $P_+\mathcal{A} = P_+\mathcal{A}P_+$. Altogether, we have proved that $\mathcal{A}P_+ = P_+\mathcal{A}$. Since $\mathcal{J} = 2P_+ - I$, the commutation relation (2.9.18) follows.

Next we show that $\mathcal{H}_1$ is maximal $\mathcal{J}$-nonnegative. Since $K$ is contractive, the definition of $\mathcal{J}$ and (2.9.19) imply that, for $x \in \mathcal{H}_1$, $x \neq 0$,

$$
\begin{aligned}
\left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right]_{\mathcal{J}} &= \big((I+K^*K)^{-1}x, x\big) - \big((I+KK^*)^{-1}Kx, Kx\big) \\
&= \big\|(I+K^*K)^{-\frac{1}{2}}x\big\|^2 - \big\|K(I+K^*K)^{-\frac{1}{2}}x\big\|^2 \geq 0.
\end{aligned}
\tag{2.9.20}
$$

Thus $\mathcal{H}_1 \oplus \{0\}$ is $\mathcal{J}$-nonnegative. Similarly, one can show that $\{0\} \oplus \mathcal{H}_2$ is $\mathcal{J}$-nonpositive. Now Lemma 2.7.3 yields that $\mathcal{H}_1 \oplus \{0\}$ is maximal $\mathcal{J}$-nonnegative.

Conversely, let $\mathcal{J}$ be a self-adjoint involution with the properties as in the theorem. If we define

$$
P_\pm := \frac{1}{2}\,(I \pm \mathcal{J})\,,
$$

then $P_+$, $P_-$ are complementary orthogonal projections with $\mathcal{J} = P_+ - P_-$. The commutation relation (2.9.18) implies that the closed subspaces $\mathcal{L}_\pm := R(P_\pm)$ are $\mathcal{A}$-invariant. If $P_1$, $P_2$ are the orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_1 \oplus \{0\}$ and $\{0\} \oplus \mathcal{H}_2$, respectively, then the assumption that $\mathcal{H}_1 \oplus \{0\}$ is $\mathcal{J}$-nonnegative shows that, for arbitrary $\mathbf{x} \in \mathcal{H}$,

$$
\big(P_+P_1\mathbf{x}, P_1\mathbf{x}\big) = \frac{1}{2}\big(P_1\mathbf{x}, P_1\mathbf{x}\big) + \frac{1}{2}\,[P_1\mathbf{x}, P_1\mathbf{x}]_{\mathcal{J}} \geq \frac{1}{2}\big\|P_1\mathbf{x}\big\|^2. \tag{2.9.21}
$$

Next we show that there exists an operator $Q \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that

$$
\mathcal{L}_- = \left\{ \begin{pmatrix} Qy \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\}. \tag{2.9.22}
$$

To this end, let $\big(\mathbf{x}_n\big)_1^\infty \subset \mathcal{L}_-$ be a sequence with $P_2\mathbf{x}_n \to 0$, $n \to \infty$. Then

$$
P_+P_1\mathbf{x}_n = P_+\big(\mathbf{x}_n - P_2\mathbf{x}_n\big) = -P_+P_2\mathbf{x}_n \longrightarrow 0, \quad n \to \infty.
$$

This and the inequality (2.9.21) imply that $P_1\mathbf{x}_n \to 0$, $n\to\infty$. This shows that $\mathcal{L}_-$ is of the form (2.9.22) with a bounded linear operator $Q$ defined on some closed subset $\mathcal{D}(Q)\subset\mathcal{H}_2$. If there were a $z\in\mathcal{D}(Q)^\perp$, $z\neq 0$, then $(0\ z)^{\rm t}\in\mathcal{L}_-^\perp=\mathcal{L}_+$ and hence

$$\left[\begin{pmatrix}0\\z\end{pmatrix},\begin{pmatrix}0\\z\end{pmatrix}\right]_{\mathcal{J}}=\left(P_+\begin{pmatrix}0\\z\end{pmatrix},P_+\begin{pmatrix}0\\z\end{pmatrix}\right)\geq 0;$$

moreover, for arbitrary $x\in\mathcal{H}_1$, we would have

$$\left[\begin{pmatrix}x\\0\end{pmatrix},\begin{pmatrix}0\\z\end{pmatrix}\right]_{\mathcal{J}}=\left(P_+\begin{pmatrix}x\\0\end{pmatrix},\begin{pmatrix}0\\z\end{pmatrix}\right)-\left(P_-\begin{pmatrix}x\\0\end{pmatrix},\begin{pmatrix}0\\z\end{pmatrix}\right)=\left(\begin{pmatrix}x\\0\end{pmatrix},\begin{pmatrix}0\\z\end{pmatrix}\right)=0.$$

From this it would follow that the subspace $\mathcal{H}_1\oplus\mathcal{D}(Q)^\perp \supsetneqq \mathcal{H}_1\oplus\{0\}$ is $\mathcal{J}$-nonnegative, a contradiction to the maximality of $\mathcal{H}_1\oplus\{0\}$. This proves that $\mathcal{D}(Q)=\mathcal{H}_2$. If we set $K:=-Q^*$, then

$$\mathcal{L}_+=\mathcal{L}_-^\perp=\left\{\begin{pmatrix}x\\-Q^*x\end{pmatrix}: x\in\mathcal{H}_1\right\}=\mathcal{G}(K).$$

Since $\mathcal{L}_+$ is $\mathcal{A}$-invariant, $K$ is a solution of the Riccati equation (2.9.17) by Proposition 2.9.12.

It remains to be proved that $K$ is a contraction. To this end, we consider the operator

$$P_1|_{\mathcal{L}_+}:\mathcal{L}_+\to\widehat{\mathcal{H}}_1\oplus\{0\},\quad P_1|_{\mathcal{L}_+}\begin{pmatrix}x\\Kx\end{pmatrix}=\begin{pmatrix}x\\0\end{pmatrix},\quad x\in\mathcal{H}_1,$$

with $\widehat{\mathcal{H}}_1:=\big(\mathcal{H}_1,\big((I+K^*K)\,\cdot,\cdot\big)\big)$. Then $P_1|_{\mathcal{L}_+}$ is invertible with $(P_1|_{\mathcal{L}_+})^{-1}=(P_1|_{\mathcal{L}_+})^*$. Hence we have $P_+\big(P_1|_{\mathcal{L}_+}\big)=\big(P_1|_{\mathcal{L}_+}\big)^*\big(P_1|_{\mathcal{L}_+}\big)P_+\big(P_1|_{\mathcal{L}_+}\big)$. This and inequality (2.9.21) yield that, for all $\mathbf{x}\in\mathcal{L}_+$,

$$\begin{aligned}\big(P_+\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x},\mathbf{x}\big)&=\big((I+K^*K)\big(P_1|_{\mathcal{L}_+}\big)P_+\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x},\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x}\big)\\&\geq\big(\big(P_1|_{\mathcal{L}_+}\big)P_+\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x},\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x}\big)\\&=\big(P_+\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x},\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x}\big)\\&\geq\frac12\big(\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x},\big(P_1|_{\mathcal{L}_+}\big)\mathbf{x}\big)\\&=\frac12(x,x).\end{aligned}$$

For $\mathbf{x}=P_+\mathbf{y}$ with $\mathbf{y}\in\mathcal{H}$, this implies

$$\big\|P_1|_{\mathcal{L}_+}P_+\mathbf{y}\big\|^2=\big(P_1|_{\mathcal{L}_+}P_+\mathbf{y},P_+\mathbf{y}\big)\geq\frac12\,\|P_+\mathbf{y}\|^2.$$

Therefore, for every $x\in\mathcal{H}_1$, $x\neq 0$,

$$\|x\|^2=\left\|P_1|_{\mathcal{L}_+}P_+\begin{pmatrix}x\\Kx\end{pmatrix}\right\|^2\geq\frac12\|x\|^2+\frac12\|Kx\|^2, \tag{2.9.23}$$

and hence $\|Kx\|^2\leq\|x\|^2$.

If $K$ is a strict (uniform, respectively) contraction, then the inequality in (2.9.20) is strict (uniform, respectively), and hence $\mathcal{H}_1 \oplus \{0\}$ is maximal $\mathcal{J}$-positive (maximal uniformly $\mathcal{J}$-positive, respectively). For the converse statement, we observe that the inequality in (2.9.21) and hence the second inequality in (2.9.23) are both strict (uniform, respectively) if $\mathcal{H}_1 \oplus \{0\}$ is maximal $\mathcal{J}$-positive (maximal uniformly $\mathcal{J}$-positive, respectively). □

In general, the Riccati equation (2.9.17) may have many contractive solutions. For example, it is easy to see that if a solution $K$ of (2.9.17) is invertible, then also $-K^{-*}$ is a solution; in particular, if $K$ is unitary, then also $-K$ is a solution. The following theorem describes all unitary solutions of the Riccati equation (2.9.17) in the case that the block operator matrix $\mathcal{A}$ is bounded.

**Theorem 2.9.15** *Let $A$, $B$, and $D$ be bounded, $A = A^*$, $D = D^*$. Then the Riccati equation* (2.9.17) *has a unitary solution if and only if there exist a unitary operator $K_0 \in L(\mathcal{H}_1, \mathcal{H}_2)$ and a self-adjoint operator $G \in L(\mathcal{H}_1)$ with*

$$K_0^* D K_0 = A, \quad B = G K_0^*. \tag{2.9.24}$$

*If* (2.9.24) *holds, then $K$ is a unitary solution of* (2.9.17) *if and only if*

$$K = K_0 U$$

*where $U \in L(\mathcal{H}_1)$ is unitary and such that*

$$AU = UA, \quad UG = GU^*. \tag{2.9.25}$$

***Proof.*** Let $K_0$ be a unitary solution of (2.9.17). Multiplying both sides of (2.9.17) from the left by the unitary operator $K_0^*$ we obtain

$$A - K_0^* D K_0 = K_0^* B^* - B K_0. \tag{2.9.26}$$

The operator on the left hand side of (2.9.26) is self-adjoint, whereas the one on the right hand side is anti-self-adjoint. Thus both sides must vanish and so (2.9.24) holds with $G = K_0^* B^*$. Vice versa, (2.9.24) clearly implies (2.9.26). Multiplication of (2.9.26) by $K_0$ from the left yields the Riccati equation for $K_0$.

If $K_0$ is a unitary operator with (2.9.24) and $U$ is as in the theorem, it is easy to check that, along with $K_0$, also $K_0 U$ satisfies (2.9.17). Vice versa, let $K$ be an arbitrary unitary solution of (2.9.17). Set $U := K_0^* K$. Evidently, $U \in L(\mathcal{H}_1)$ and $U$ is unitary. Multiplying the Riccati equation (2.9.17) for $K$ from the left by $K_0^*$ and from the right by $U^*$, we obtain, using (2.9.24),

$$UAU^* - A = GU^* - UG.$$

Now the same reasoning as in the first part of the proof shows that $U$ satisfies (2.9.25). □

**Remark 2.9.16** Another description of the set of all possible contractive solutions of a Riccati equation (2.9.17) was given by V. Kostrykin, A. Makarov, and A.K. Motovilov (see [KMM03a, Theorem 6.2]); their assumption is that there exists a contractive solution $K_0$ whose graph is a spectral subspace for the corresponding block operator matrix. It is shown that the latter holds if and only if the solution $K_0$ is an isolated point (in the operator norm topology) in the set of all solutions of the Riccati equation; this geometric approach is based on a result of R.G. Douglas and C. Pearcy on invariant subspaces of normal operators (see [DP68]).

In order to establish a uniqueness result for strictly contractive solutions of Riccati equations (2.9.17), we need the following notions.

**Definition 2.9.17** Let $T$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with spectral function $E_T$. A $T$-invariant subspace $\mathcal{L} \subset \mathcal{H}$ is called

i) *$T$-spectral* if there exists a closed subset $\Delta \subset \mathbb{R}$ with $\mathcal{L} = E_T(\Delta)\mathcal{H}$,
ii) *$T$-normal* if every $T$-reducing subspace either belongs to $\mathcal{L}$ or contains non-zero vectors that are orthogonal to $\mathcal{L}$.

**Lemma 2.9.18** *For self-adjoint $T$, every $T$-spectral subspace is $T$-normal.*

***Proof.*** Let $\mathcal{L}$ be $T$-spectral, $\mathcal{L} = E_T(\Delta)\mathcal{H}$ for some closed subset $\Delta \subset \mathbb{R}$. If $\mathcal{L}_0$ is a $T$-reducing subspace, then $T$ is block diagonal with respect to the decomposition $\mathcal{H} = \mathcal{L}_0 \oplus \mathcal{L}_0^\perp$. Hence, for the restriction $T_0 := T|_{\mathcal{L}_0}$, we have $\sigma(T_0) \subset \sigma(T)$. If $\sigma(T_0) \subset \Delta$, then the spectral theorem for self-adjoint operators yields $\mathcal{L}_0 \subset \mathcal{L}$. If there exists a $\lambda_0 \in \sigma(T_0) \setminus \Delta$, we can choose $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap \Delta = \emptyset$. Then the range of $E_{T_0}\big((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)\big)$ is nontrivial and orthogonal to $\mathcal{L}$. □

**Proposition 2.9.19** *Let $A = A^*$, $D = D^*$, and let $B$, $D$ be bounded. If $K$ is a strictly contractive solution of the Riccati equation* (2.9.17) *with $\mathcal{A}$-normal graph $\mathcal{G}(K)$, then $K$ is the unique contractive solution of* (2.9.17).

***Proof.*** Let $K$ be as above and let $K_1$ be another contractive solution of (2.9.17). Then the graphs $\mathcal{G}(K)$, $\mathcal{G}(K_1)$ are $\mathcal{A}$-reducing by Proposition 2.9.12 and Remark 2.9.13. If there exists a nonzero $\mathbf{x} \in \mathcal{G}(K_1) \cap \mathcal{G}(K)^\perp$, then there exist $x \in \mathcal{H}_1, y \in \mathcal{H}_2$, $x, y \neq 0$, with

$$\mathbf{x} = \begin{pmatrix} x \\ K_1 x \end{pmatrix} = \begin{pmatrix} -K^* y \\ y \end{pmatrix}.$$

Because $K$ and hence $K^*$ are strict contractions, we obtain the contradiction

$$\|x\| \geq \|K_1 x\| = \|y\| > \|K^* y\| = \|x\| . \tag{2.9.27}$$

By assumption, $\mathcal{G}(K)$ is $\mathcal{A}$-normal and so $\mathcal{G}(K_1) \subset \mathcal{G}(K)$. Since both subspaces are maximal $\mathcal{J}$-nonnegative by Theorem 2.9.14, they coincide. □

The following uniqueness result can be obtained from Theorem 2.9.14 and Proposition 2.9.19 (see [ALT01, Theorem 6.3] and the generalization in [KMM04, Theorem 4.1]).

**Theorem 2.9.20** *Let $A = A^*$, $D = D^*$ with $\max \sigma(D) \leq \min \sigma(A)$, and let $B$, $D$ be bounded. Then the Riccati equation* (2.9.17) *has a unique contractive solution $K$, and this solution is strictly contractive.*

***Proof.*** By Proposition 2.7.13, $\ker \mathcal{A}$ has the kernel splitting property. Accordingly, we define $\mathcal{L}_+$ as in Remark 2.7.12, let $P_+$ be the orthogonal projection onto $\mathcal{L}_+$, and set $P_- := I - P_+$. Then $\mathcal{J} := P_+ - P_-$ is a self-adjoint involution commuting with $\mathcal{A}$. Using a similar reasoning as in the proof of Theorem 2.7.7 and Remark 2.7.12 (see the proof of [ALT01, Theorem 6.2]), we can show that $\mathcal{H}_1$ is maximal $\mathcal{J}$-positive. Now all claims follow from Theorem 2.9.14 and Proposition 2.9.19. □

**Remark 2.9.21** Theorem 2.9.20 can be used to show the uniqueness of the solutions of Riccati equations established in some earlier results:

i) If $\max \sigma(D) < \min \sigma(A)$ in Theorem 2.9.20, then the solution $K$ of the Riccati equation established in [AL95, Theorem 2.3] (and hence the one in Theorem 2.7.7 and its Corollary 2.7.8) is unique.

ii) If $A$ and $-D$ are not self-adjoint, but regularly m-accretive with $\operatorname{Re} W(D) < 0 < \operatorname{Re} W(A)$ and if $B$, $D$ are bounded, then the solutions $K_+$, $K_-$ of the Riccati equations derived in [LT98, Theorem 4.1], which are contractions by Theorem 2.7.21 and its Corollary 2.7.23 (see also Remark 2.7.22), are unique.

**Corollary 2.9.22** *Let $A = A^*$, $D = D^*$, and let $B, D$ be bounded. If there exists a bounded interval $(a, b) \subset \mathbb{R}$ such that $\sigma(D) \subset (a, b)$, $\sigma(A) \cap [a, b] = \emptyset$, $S_1(a)$, $S_2(a)$ are bijective and $I - (b-a) S_1(a)^{-1} > 0$, $I - (b-a) S_2(a)^{-1} < 0$, then the Riccati equation* (2.9.17) *has a unique strictly contractive solution.*

***Proof.*** The claim follows from Theorem 2.9.20 applied to the block operator matrix $\widetilde{\mathcal{A}} := (\mathcal{A}-b)(\mathcal{A}-a)^{-1} = I-(b-a)(\mathcal{A}-a)^{-1}$. □

We close this section by mentioning an existence and uniqueness result for accretive solutions of Riccati equations, which can be deduced from Theorem 2.9.14 and Proposition 2.9.19 by a suitable transformation (see [ALT01, Theorem 7.2]).

**Remark 2.9.23** Let $\mathcal{H}_1 = \mathcal{H}_2$ and $V, Q, R \in L(\mathcal{H}_1)$. If there exist constants $\gamma \in \mathbb{R}$ and $c > 0$ such that

$$\mathrm{Re}\,(\gamma^2 Q) = \mathrm{Re}\,(R) \geq c,$$

then the Riccati equation

$$YQY + YV + V^*Y - R = 0$$

has a unique accretive solution $Y \in L(\mathcal{H}_1)$ which is strictly accretive. If, additionally, $Q = Q^*$ and $R = R^*$, then $Y$ is strictly positive.

***Proof.*** For $\gamma > 0$, the relations

$$Y = \gamma(I+K)(I-K)^{-1}, \quad K = (Y-\gamma I)(Y+\gamma I)^{-1}$$

establish a bijective correspondence between all strictly accretive operators $Y \in L(\mathcal{H}_1)$ and all strict contractions $K \in L(\mathcal{H}_1)$. Now the claim follows from Remark 2.9.21 ii) applied to the block operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \gamma I & I \\ -\gamma I & I \end{pmatrix} \begin{pmatrix} V & Q \\ R & -V^* \end{pmatrix} \begin{pmatrix} I & -I \\ \gamma I & \gamma I \end{pmatrix}. \qquad \square$$

**Remark 2.9.24** The fixed point approach is not restricted to the case $\mathrm{dist}\big(\sigma(A), \sigma(D)\big) > 0$. In a series of papers by R. Mennicken, A.K. Motovilov, and V. Hardt, it was also used in the case where $\sigma(D)$ is partly or entirely embedded in $\sigma_{\mathrm{c}}(A)$. The crucial assumption there is that the second Schur complement admits an analytic continuation through the cuts along the branches of the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(A)$; the latter is ensured by conditions on $B$. The diagonally dominant self-adjoint case (with unbounded $A$, $D$ and bounded $B$) was treated in [MM98], [MM99]. The upper dominant self-adjoint case with semi-bounded $A$, unbounded $B$ and bounded $D$ was studied in [HMM02]; in [HMM03] an upper dominant non-self-adjoint situation was considered where it was assumed that only $A$ is self-adjoint and semi-bounded, $B$, $C$ are unbounded so that $C(A-z)^{-1}B$ is bounded, and $D$ is bounded. As in Section 2.8, the results were used to prove theorems on block diagonalization and half range basisness.

## 2.10 Variational principles

In this section we generalize the variational principles and eigenvalue estimates established in Section 1.10 not only to unbounded, but also to $\mathcal{J}$-self-adjoint block operator matrices. Moreover, we give a different proof which is based on variational principles for unbounded operator functions by D. Eschwé and M. Langer applied to the Schur complements (see [EL04]). The crucial assumption here is a certain monotonicity of the Schur complements which we ensure by assuming that the diagonal entries are bounded from above and below, respectively. Another difference to the preceding part of this chapter is that we introduce the Schur complements by means of quadratic forms, thus allowing for larger classes of block operator matrices; in particular, at least for bounded diagonal entries, we can also treat the off-diagonally dominant case.

In the following we always assume that $A$, $D$ are self-adjoint and semibounded, $A \geq a = \min \sigma(A)$, $D \leq d = \max \sigma(D)$, and that either $C = B^*$ or $C = -B^*$. We formulate and prove all results for eigenvalues to the right of $d$ in terms of the functionals $\lambda_+$ related to the quadratic numerical range; analogous results hold for eigenvalues to the left of $a$ if $\lambda_+$ is replaced by $\lambda_-$.

A quadratic form $\mathfrak{t}$ with domain $\mathcal{D}(\mathfrak{t})$ in a Hilbert space is called *regularly quasi-accretive* if for its numerical range $W(\mathfrak{t}) := \big\{ \mathfrak{t}[x] : x \in \mathcal{D}(\mathfrak{t}), \|x\| = 1 \big\}$ there exist $\alpha \in \mathbb{R}$ and $\vartheta \in [0, \pi/2)$ such that

$$W(\mathfrak{t}) \subset \alpha + \Sigma_\vartheta, \quad \Sigma_\vartheta := \big\{ r e^{\mathrm{i}\phi} \colon r \geq 0, |\phi| \leq \vartheta \big\}.$$

A family $\mathfrak{t}(\lambda)$, $\lambda \in \Omega \subset \mathbb{C}$, of quadratic forms in a Hilbert space is called *holomorphic of type* (a) if, for every $\lambda \in \Omega$, the form $\mathfrak{t}(\lambda)$ is regularly quasi-accretive, closed with dense $\lambda$-independent domain $\mathcal{D}\big(\mathfrak{t}(\lambda)\big) = \mathfrak{D}$, and the function $\mathfrak{t}(\cdot)[x]$ is holomorphic in $\Omega$. The corresponding family $T(\lambda)$, $\lambda \in \Omega$, of regularly quasi-accretive operators induced by the first representation theorem is called *holomorphic of type* (B) (see [Kat95, Section VII.4.2]).

Let $\mathfrak{a}$ and $\mathfrak{d}$ be the closures of the quadratic forms induced by $A$ and $D$, respectively, with domains $\mathcal{D}(\mathfrak{a}) = \mathcal{D}(|A|^{1/2})$, $\mathcal{D}(\mathfrak{d}) = \mathcal{D}(|D|^{1/2})$ and

$$\mathfrak{a}[x,y] = (Ax, y), \quad \mathfrak{a}[x] = (Ax, x), \quad x \in \mathcal{D}(A),\ y \in \mathcal{D}(|A|^{1/2}),$$

$$\mathfrak{d}[x,y] = (Dx, y), \quad \mathfrak{d}[x] = (Dx, x), \quad x \in \mathcal{D}(D),\ y \in \mathcal{D}(|D|^{1/2}),$$

(see [EE87, Section IV.4] and note that regularly quasi-accretive forms are called sectorial therein). We begin by considering (essentially) self-adjoint block operator matrices under three different domain assumptions. For all three cases, we introduce the sets

$$\mathfrak{D}_1 := \mathcal{D}(|A|^{1/2}) \cap \mathcal{D}(B^*), \quad \mathfrak{D}_2 := \mathcal{D}(|D|^{1/2}). \tag{2.10.1}$$

**Proposition 2.10.1** *Assume that $A = A^*$ is bounded from below, $D = D^*$ is bounded from above, $d := \max \sigma(D)$, $B$ is closed, $C = B^*$, and let one of the following conditions be satisfied:*

(a) $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$, $\mathcal{D}(|D|^{1/2}) \subset \mathcal{D}(B)$;
(b) $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$, $\mathcal{D}(B) \subset \mathcal{D}(D)$ *and* $\mathcal{D}(B)$ *is a core of* $D$;
(c) $A$, $D$ *are bounded.*

*Then $\mathcal{A}$ is diagonally dominant and self-adjoint in case* (a), *upper dominant and essentially self-adjoint in case* (b), *off-diagonally dominant and self-adjoint in case* (c), *and in all cases* $\mathcal{D}(\overline{\mathcal{A}}) \subset \mathfrak{D}_1 \oplus \mathfrak{D}_2$. *Moreover,*

i) *the quadratic form $\mathfrak{s}_1(\lambda)$, defined for $\lambda \in \mathbb{C}_+^d := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > d\}$ by*
$$\mathfrak{s}_1(\lambda)[x,y] := \mathfrak{a}[x,y] - \lambda(x,y) - \big((D-\lambda)^{-1}B^*x, B^*y\big), \quad x,y \in \mathfrak{D}_1,$$
*is closed and regularly quasi-accretive in $\mathcal{H}_1$ with domain $\mathcal{D}(\mathfrak{s}_1(\lambda)) = \mathfrak{D}_1 = \mathcal{D}(|A|^{1/2}) \cap \mathcal{D}(B^*)$ independent of $\lambda$, and $\mathfrak{s}_1$ is holomorphic on $\mathbb{C}_+^d$;*

ii) *the operator $S_1(\lambda)$ thus induced by $\mathfrak{s}_1(\lambda)$ is regularly quasi-accretive and*
$$\mathcal{D}(S_1(\lambda)) = \mathcal{D}(A) \text{ for (a)},$$
$$\mathcal{D}(S_1(\lambda)) = \Big\{x \in \mathcal{D}(|A|^{1/2}) : x - \overline{(A-\nu)^{-1}B}(D-\lambda)^{-1}B^*x \in \mathcal{D}(A)\Big\} \text{ for (b)},$$
$$\mathcal{D}(S_1(\lambda)) = \mathcal{D}\big(B(D-\lambda)^{-1}B^*\big) \text{ for (c)}, \tag{2.10.2}$$
*where, in the second case, $\nu \in (-\infty, \min\sigma(A))$ is arbitrary;*

iii) *the spectra of $\overline{\mathcal{A}}$ and $S_1$ are related by*
$$\sigma(\overline{\mathcal{A}}) \cap \mathbb{C}_+^d = \sigma(S_1) \cap \mathbb{C}_+^d, \tag{2.10.3}$$
$$\sigma_{\mathrm{p}}(\overline{\mathcal{A}}) \cap \mathbb{C}_+^d = \sigma_{\mathrm{p}}(S_1) \cap \mathbb{C}_+^d. \tag{2.10.4}$$

***Proof.*** In case (a), by Corollary 2.1.20, $B^*$ is $A$-bounded with relative bound 0 and $B$ is $D$-bounded with relative bound 0, respectively. Hence $\mathcal{A}$ is diagonally dominant of order 0 and thus self-adjoint by Theorem 2.6.6 i). In case (b), Proposition 2.3.6 shows that $\mathcal{A}$ is essentially self-adjoint. In case (c), the self-adjointness of $\mathcal{A}$ is obvious, *e.g.* from Theorem 2.6.6 iii).

In cases (a) and (b), we have $\mathfrak{D}_1 = \mathcal{D}(|A|^{1/2})$ and hence
$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus \mathcal{D}(D) = \mathcal{D}(|A|) \oplus \mathcal{D}(|D|) \subset \mathcal{D}(|A|^{1/2}) \oplus \mathcal{D}(|D|^{1/2}) = \mathfrak{D}_1 \oplus \mathfrak{D}_2.$$
In case (c), we have $\mathfrak{D}_1 = \mathcal{D}(B^*)$ and thus
$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(B^*) \oplus \mathcal{D}(B) \subset \mathcal{D}(B^*) \oplus \mathcal{D}(D) \subset \mathcal{D}(B^*) \oplus \mathcal{D}(|D|^{1/2}) = \mathfrak{D}_1 \oplus \mathfrak{D}_2.$$
For case (b), Proposition 2.3.6 shows that
$$\mathcal{D}(\overline{\mathcal{A}}) \subset \mathcal{D}(|A|^{1/2}) \oplus \mathcal{D}(D) \subset \mathcal{D}(|A|^{1/2}) \oplus \mathcal{D}(|D|^{1/2}) = \mathfrak{D}_1 \oplus \mathfrak{D}_2.$$

i) The form $\mathfrak{a}$ is closed and regularly quasi-accretive with domain $\mathcal{D}(\mathfrak{a}) = \mathcal{D}(|A|^{1/2})$, the form $\mathfrak{t}_0(\lambda)$ defined by $\big((\lambda - D)^{-1}B^*x, B^*x\big)$ is closable and regularly quasi-accretive on $\mathcal{D}(B^*)$ for $\lambda \in \mathbb{C}^d_+$, and for its closure $\mathfrak{t}(\lambda)$ we have $\mathcal{D}\big(\mathfrak{t}(\lambda)\big) \supset \mathcal{D}(B^*)$. Hence the sum $\mathfrak{s}_1(\lambda) = \mathfrak{a} - \lambda + \mathfrak{t}(\lambda)$ is closed and regularly quasi-accretive on $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}\big(\mathfrak{t}(\lambda)\big)$ for $\lambda \in \mathbb{C}^d_+$ (see [Kat95, Theorem VI.1.27] and [Kat95, Theorem VI.1.31]). In cases (a) and (b), we have $\mathcal{D}(\mathfrak{a}) = \mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*) \subset \mathcal{D}\big(\mathfrak{t}(\lambda)\big)$ and thus $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}\big(\mathfrak{t}(\lambda)\big) = \mathcal{D}(|A|^{1/2}) = \mathfrak{D}_1$. In case (c), we have $\mathcal{D}(\mathfrak{a}) = \mathcal{H}_1$ and $\mathcal{D}\big(\mathfrak{t}(\lambda)\big) = \mathcal{D}\big(\mathfrak{t}_0(\lambda)\big) = \mathcal{D}(B^*)$ since $D$ is bounded and hence $(D - \lambda)^{-1}$ is boundedly invertible for $\lambda \in \mathbb{C}^d_+$; thus $\mathcal{D}(\mathfrak{a}) \cap \mathcal{D}\big(\mathfrak{t}(\lambda)\big) = \mathcal{D}(B^*) = \mathfrak{D}_1$.

ii) In the diagonally dominant case (a), the operator $B(D - \lambda)^{-1}$ is bounded since it is closed and $\mathcal{D}(D) \subset \mathcal{D}(B)$. By Corollary 2.1.20, the assumption $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$ implies that $B^*$ is $A$-bounded with $A$-bound 0. Hence $B(D - \lambda)^{-1}B^*$ is $A$-bounded with $A$-bound 0. Since $A$ is assumed to be bounded from below, $S_1(\lambda) = A - \lambda - B(D - \lambda)^{-1}B^*$ is regularly quasi-accretive with domain $\mathcal{D}(A)$ and self-adjoint for real $\lambda \in (d, \infty)$; in particular, $S_1(\lambda)$ is closed.

To show ii) in the upper dominant case (b), we follow the lines of the proof of [EL04, Proposition 4.4]. Let $\nu \in \big(-\infty, \min\sigma(A)\big)$ be arbitrary. We fix $\lambda \in \mathbb{C}^d_+$ and define the sesquilinear form

$$\mathfrak{t}[x, y] := \mathfrak{s}_1(\lambda)[x, y] - (\nu - \lambda)\big(x, y\big), \quad x, y \in \mathcal{D}(\mathfrak{s}_1) = \mathcal{D}(|A|^{1/2}).$$

Then, by i), $\mathfrak{t}$ is closed and sectorial on the domain $\mathcal{D}(\mathfrak{t}) = \mathcal{D}(|A|^{1/2})$. Let $T$ be the operator induced by $\mathfrak{t}$. Then $T$ is regularly quasi-accretive, $\mathcal{D}(T) = \mathcal{D}\big(S_1(\lambda)\big)$, and $T = S_1(\lambda) - (\nu - \lambda)$. Because of the identity $\overline{(A - \nu)^{-1}B} = (A - \nu)^{-1/2}\overline{(A - \nu)^{-1/2}B}$, it is sufficient to prove that $T$ coincides with the operator $\widehat{T}$ given by

$$\mathcal{D}\big(\widehat{T}\big) = \big\{x \in \mathcal{D}(|A|^{1/2}) : \big((A-\nu)^{1/2} - \overline{(A-\nu)^{-1/2}B}(D-\lambda)^{-1}B^*\big)x \in \mathcal{D}(|A|^{1/2})\big\},$$

$$\widehat{T}x = (A - \nu)^{1/2}\big((A - \nu)^{1/2} - \overline{(A - \nu)^{-1/2}B}(D - \lambda)^{-1}B^*\big)x, \quad x \in \mathcal{D}\big(\widehat{T}\big).$$

To this end, let $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(|A|^{1/2})$. Then

$$\begin{aligned}
&\mathfrak{t}[x, y] \\
&= \big((A-\nu)^{1/2}x, (A-\nu)^{1/2}y\big) - \big((D-\lambda)^{-1}B^*x, B^*y\big) \\
&= \big((A-\nu)^{1/2}x, (A-\nu)^{1/2}y\big) - \big((D-\lambda)^{-1}B^*x, B^*(A-\nu)^{-1/2}(A-\nu)^{1/2}y\big) \\
&= \big((A-\nu)^{1/2}x - \overline{(A-\nu)^{-1/2}B}(D-\lambda)^{-1}B^*x, (A-\nu)^{1/2}y\big).
\end{aligned}$$

On the other hand, by the definition of $T$,

$$\mathfrak{t}[x, y] = (Tx, y) = \big(Tx, (A-\nu)^{-1/2}(A-\nu)^{1/2}y\big) = \big((A-\nu)^{-1/2}Tx, (A-\nu)^{1/2}y\big).$$

Since $\nu \in \rho(A)$, we have $\big\{(A-\nu)^{1/2}y : y \in \mathcal{D}(|A|^{1/2})\big\} = \mathcal{H}_1$ and hence

$$(A-\nu)^{-1/2}Tx = (A-\nu)^{1/2}x - \overline{(A-\nu)^{-1/2}B}(D-\lambda)^{-1}B^*x.$$

This implies that the right hand side of the latter equality belongs to $\mathcal{D}((A-\nu)^{1/2}) = \mathcal{D}(|A|^{1/2})$. Hence $x \in \mathcal{D}(\widehat{T})$ and $\widehat{T}x = Tx$.

Vice versa, suppose that $x \in \mathcal{D}(\widehat{T})$ and $y \in \mathcal{D}(|A|^{1/2})$. The same calculation as above shows that $\big(\widehat{T}x, y\big) = \mathfrak{t}[x,y]$. Then [Kat95, Theorem VI.2.1 iii)] yields $x \in \mathcal{D}(T)$ and $\widehat{T}x = Tx$.

For the proof of ii) in the off-diagonally dominant case (c), let first $x \in \mathcal{D}\big(B(D-\lambda)^{-1}B^*\big) = \big\{x \in \mathcal{D}(B^*) : (D-\lambda)^{-1}B^*x \in \mathcal{D}(B)\big\}$. Then

$$\big((\lambda-D)^{-1}B^*x, B^*y\big) = \big(B(\lambda-D)^{-1}B^*x, y\big), \quad y \in \mathcal{D}(B^*).$$

According to [Kat95, Theorem VI.2.1 iii)], this implies that $x$ belongs to the domain of the operator induced by the form $\big((\lambda-D)^{-1}B^*x, B^*x\big)$, and hence $x \in \mathcal{D}\big(S_1(\lambda)\big)$. To prove the converse inclusion, let $x \in \mathcal{D}\big(S_1(\lambda)\big) \subset \mathcal{D}\big(\mathfrak{s}_1(\lambda)\big) = \mathfrak{D}_1$. Then $x \in \mathcal{D}(B^*)$ and the form $\mathfrak{s}_1(\lambda)[x,y] = \big(S_1(\lambda)x, y\big)$ is bounded in $y$, which shows that $(D-\lambda)^{-1}B^*x \in \mathcal{D}(B)$.

iii) In the diagonally dominant case (a), the assertion about the spectra follows from Theorem 2.3.3 ii) since $S_1(\lambda)$ is closed and the condition $\mathcal{D}(D^*) = \mathcal{D}(D) \subset \mathcal{D}(B)$ is satisfied (which is not true in the other two cases) and hence $(D-\lambda)^{-1}B^*$ is bounded on $\mathcal{D}(B^*)$ by Remark 2.2.19.

The proof of iii) in the upper dominant case (b) again follows the lines of the proof of [EL04, Proposition 4.4]. For the inclusion "$\supset$" in (2.10.3), let $\lambda \in \rho\big(\overline{\mathcal{A}}\big) \cap \mathbb{C}^d_+$. Then, for arbitrary $f \in \mathcal{H}_1$, there exists a unique $(x\ y)^{\mathrm{t}} \in \mathcal{D}\big(\overline{\mathcal{A}}\big)$, *i.e.* $x \in \mathcal{D}(|A|^{1/2})$, $y \in \mathcal{D}(D)$ with $x + \overline{(A-\nu)^{-1}B}y \in \mathcal{D}(A)$ (see Proposition 2.3.6), such that $(\overline{\mathcal{A}}-\lambda)(x\ y)^{\mathrm{t}} = (f\ 0)^{\mathrm{t}}$, or, equivalently,

$$A\big(x + \overline{(A-\nu)^{-1}B}y\big) - \nu\overline{(A-\nu)^{-1}B}y - \lambda x = f,$$
$$B^*x + (D-\lambda)y = 0.$$

Since $\lambda \in \mathbb{C}^d_+ \subset \rho(D)$, the second relation shows that $y = -(D-\lambda)^{-1}B^*x$; in particular, $(x\ y)^{\mathrm{t}} \neq 0$ implies that $x \neq 0$. Using this relation for $y$, we obtain $x - \overline{(A-\nu)^{-1}B}(D-\lambda)^{-1}B^*x = x + \overline{(A-\nu)^{-1}B}y \in \mathcal{D}(A)$ and

$$A\big(x - \overline{(A-\nu)^{-1}B}(D-\lambda)^{-1}B^*x\big) - \nu\big(x - \overline{(A-\nu)^{-1}B}(D-\lambda)^{-1}B^*x\big) + (\nu-\lambda)x = f.$$

By ii), this means that $x \in \mathcal{D}\big(S_1(\lambda)\big)$ and $S_1(\lambda)x = f$. Since $(x\ y)^{\mathrm{t}}$ is uniquely determined by $f$ and $y$ is uniquely determined by $x$, this proves that $\lambda \in \rho(S_1) \cap \mathbb{C}^d_+$.

For the inclusion "$\subset$" in (2.10.4), we use the same reasoning as above with $f = 0$. For the inclusion "$\supset$" in (2.10.4), let $\lambda \in \sigma_{\mathrm{p}}(S_1) \cap \mathbb{C}^d_+$ and let $x \in \ker S_1(\lambda)$, $x \neq 0$, be a corresponding eigenvector. If we set $y := -(D-\lambda)^{-1}B^*x$, then $(x\,y)^{\mathrm{t}} \in \mathcal{D}(\overline{\mathcal{A}})$, $(x\,y)^{\mathrm{t}} \neq 0$, and $\big(\overline{\mathcal{A}}-\lambda\big)(x\,y)^{\mathrm{t}} = (0\,0)^{\mathrm{t}}$.

In case (c), for the inclusion "$\supset$" in (2.10.3) and for "$=$" (2.10.4), we observe that the solutions of

$$(\mathcal{A}-\lambda)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{aligned} (A-\lambda)x + By &= f, \\ B^*x + (D-\lambda)y &= 0, \end{aligned}$$

with $x \in \mathcal{D}(B^*)$, $y \in \mathcal{D}(B)$, and arbitrary $f \in \mathcal{H}_1$ are in one-to-one correspondence to the solutions of $S_1(\lambda)x = f$, *i.e.*

$$(A-\lambda)x - B(D-\lambda)^{-1}B^*x = f,$$

with $x \in \mathcal{D}\big(S_1(\lambda)\big)$ via the relation $y = -(D-\lambda)^{-1}B^*x$.

It remains to show the inclusion "$\subset$" in (2.10.3) in cases (b) and (c). Let $\lambda \in \rho(S_1) \cap \mathbb{C}^d_+$. We may assume that $\lambda \in \mathbb{R}$; otherwise, $\lambda \in \rho\big(\overline{\mathcal{A}}\big)$ since $\overline{\mathcal{A}}$ is self-adjoint. It is not difficult to see that, on the set $\mathcal{H}_1 \oplus \mathcal{D}\big((B(D-\lambda)^{-1}\big)$, the (formal) inverse $\mathcal{R}(\lambda)$ of $\mathcal{A}-\lambda$ is given by

$$\begin{pmatrix} S_1(\lambda)^{-1} & -S_1(\lambda)^{-1}B(D-\lambda)^{-1} \\ -(D-\lambda)^{-1}B^*S_1(\lambda)^{-1} & (D-\lambda)^{-1} + (D-\lambda)^{-1}B^*S_1(\lambda)^{-1}B(D-\lambda)^{-1} \end{pmatrix}$$

(compare Theorem 2.3.3 ii)). It remains to be shown that in both cases the set $\mathcal{H}_1 \oplus \mathcal{D}\big(B(D-\lambda)^{-1}\big)$ is dense in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and that $\mathcal{R}(\lambda)$ is bounded.

In case (b), the assumption that $\mathcal{D}(D)$ is a core of $B$ implies that $\mathcal{D}\big(B(D-\lambda)^{-1}\big)$ is dense in $\mathcal{H}_2$; in case (c), the fact that $D$ is bounded and $\mathcal{D}(B)$ is dense shows that $\mathcal{D}\big(B(D-\lambda)^{-1}\big) = (D-\lambda)\big(\mathcal{D}(B)\big)$ is dense in $\mathcal{H}_2$. Further, in case (b) the operator $|S_1(\lambda)|^{1/2}$ has domain $\mathcal{D}\big(\mathfrak{s}_1(\lambda)\big) = \mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$; in case (c), $|S_1(\lambda)|^{1/2}$ has domain $\mathcal{D}\big(\mathfrak{s}_1(\lambda)\big) = \mathcal{D}(B^*)$. So $(D-\lambda)^{-1}B^*|S_1(\lambda)|^{-1/2}$ is everywhere defined and closable in case (b), closed in case (c), and thus bounded in both cases. This implies that $|S_1(\lambda)|^{-1/2}B(D-\lambda)^{-1} \subset \big((D-\lambda)^{-1}B^*|S_1(\lambda)|^{-1/2}\big)^*$ is bounded and hence so is

$$\begin{aligned} &(D-\lambda)^{-1}B^*S_1(\lambda)^{-1}B(D-\lambda)^{-1} \\ &= (D-\lambda)^{-1}B^*|S_1(\lambda)|^{-1/2}\,\mathrm{sign}(S_1(\lambda)^{-1})|S_1(\lambda)|^{-1/2}B(D-\lambda)^{-1}. \end{aligned}$$

In a similar way, one can show that the off-diagonal elements of the resolvent are bounded. This completes the proof that $\lambda \in \rho\big(\overline{\mathcal{A}}\big) \cap \mathbb{C}^d_+$. □

**Corollary 2.10.2** *Under the assumptions of Proposition* 2.10.1*, the family* $\mathfrak{s}_1(\lambda)$*,* $\lambda \in \mathbb{C}^d_+$*, is holomorphic of type* (a) *and the Schur complements* $S_1(\lambda)$*,* $\lambda \in \mathbb{C}^d_+$*, form a holomorphic family of type* (B)*.*

The variational principles proved below use the functional $\lambda_+$ related to the quadratic numerical range of $\mathcal{A}$ as well as the classical Rayleigh functional $(\mathcal{A}\,\cdot,\cdot)$. To this end, we first extend both functionals from the domain $\mathcal{D}(\mathcal{A}) = \big(\mathcal{D}(A)\cap\mathcal{D}(B^*)\big)\oplus\big(\mathcal{D}(B)\cap\mathcal{D}(D)\big)$ to the set $\mathfrak{D}_1\oplus\mathfrak{D}_2 = \big(\mathcal{D}(|A|^{1/2})\cap\mathcal{D}(B^*)\big)\oplus\mathcal{D}(|D|^{1/2})$.

**Definition 2.10.3** For $x\in\mathfrak{D}_1$, $y\in\mathfrak{D}_2$, we define

$$\mathfrak{A}\left[\begin{pmatrix}x\\y\end{pmatrix}\right] := \mathfrak{a}[x] + (B^*x,y) + (y,B^*x) + \mathfrak{d}[y],$$

and, if $x,y\neq 0$,

$$\lambda_+\begin{pmatrix}x\\y\end{pmatrix} := \frac{1}{2}\left(\frac{\mathfrak{a}[x]}{\|x\|^2}+\frac{\mathfrak{d}[y]}{\|y\|^2}+\sqrt{\left(\frac{\mathfrak{a}[x]}{\|x\|^2}-\frac{\mathfrak{d}[y]}{\|y\|^2}\right)^2+4\frac{|(B^*x,y)|^2}{\|x\|^2\,\|y\|^2}}\right). \quad (2.10.5)$$

The following variational characterizations for eigenvalues were established in [KLT04, Theorem 3.1]; for the special case of diagonally dominant block operator matrices with bounded off-diagonal entries they were first proved in [LLT02]. Throughout this section, $\mathcal{L}$ always denotes a finite-dimensional subspace of the first component $\mathfrak{D}_1\subset\mathcal{H}_1$.

**Theorem 2.10.4** *Suppose that $A=A^*$ is bounded from below, $D=D^*$ is bounded from above, $d:=\max\sigma(D)$, $B$ is closed, $C=B^*$, and that the block operator matrix $\mathcal{A}$ satisfies one of the conditions* (a), (b), *or* (c) *in Proposition* 2.10.1 (*so that $\mathcal{A}=\overline{\mathcal{A}}$ is self-adjoint in cases* (a) *and* (c), *and $\mathcal{A}$ is essentially self-adjoint in case* (b)). *Set*

$$\lambda_{\mathrm{e}} := \min\big(\sigma_{\mathrm{ess}}(\overline{\mathcal{A}})\cap(d,\infty)\big) \quad (2.10.6)$$

*and*

$$\kappa_-(\lambda) := \dim\mathcal{L}_{(-\infty,0)}\big(S(\lambda)\big), \quad \lambda\in\rho(D)\cap\mathbb{R}.$$

*Further, assume that there exists a point $\gamma\in(d,\infty)$ such that $\kappa_-(\gamma)<\infty$. Then there exists an $\alpha>d$ so that $(d,\alpha)\subset\rho\big(\overline{\mathcal{A}}\big)$. If we set $\kappa:=\kappa_-(\alpha)$ $(<\infty)$ and let $\lambda_1\le\lambda_2\le\cdots\le\lambda_N$, $N\in\mathbb{N}_0\cup\{\infty\}$, be the finite or infinite sequence of the eigenvalues of $\overline{\mathcal{A}}$ in the interval $(d,\lambda_{\mathrm{e}})$ counted with multiplicities, then, for $n=1,2,\dots,N$,*

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathfrak{D}_1\\ \dim\mathcal{L}=\kappa+n}}\ \max_{\substack{x\in\mathcal{L}\\ x\neq0}}\ \max_{\substack{y\in\mathfrak{D}_2\\ y\neq0}}\ \lambda_+\begin{pmatrix}x\\y\end{pmatrix} = \max_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=\kappa+n-1}}\ \inf_{\substack{x\in\mathfrak{D}_1,\,x\neq0\\ x\perp\mathcal{L}}}\ \max_{\substack{y\in\mathfrak{D}_2\\ y\neq0}}\ \lambda_+\begin{pmatrix}x\\y\end{pmatrix} \quad (2.10.7)$$

$$= \min_{\substack{\mathcal{L}\subset\mathfrak{D}_1\\ \dim\mathcal{L}=\kappa+n}}\ \max_{\substack{x\in\mathcal{L}\\ x\neq0}}\ \max_{y\in\mathfrak{D}_2}\ \frac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2} = \max_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=\kappa+n-1}}\ \inf_{\substack{x\in\mathfrak{D}_1,\,x\neq0\\ x\perp\mathcal{L}}}\ \max_{y\in\mathfrak{D}_2}\ \frac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2}. \quad (2.10.8)$$

*If* $\mu_m$ *denotes any of the four expressions*

$$\begin{array}{ll} \inf\limits_{\substack{\mathcal{L}\subset\mathfrak{D}_1\\ \dim\mathcal{L}=m}} \max\limits_{\substack{x\in\mathcal{L}\\ x\neq 0}} \sup\limits_{\substack{y\in\mathfrak{D}_2\\ y\neq 0}} \lambda_+\begin{pmatrix}x\\y\end{pmatrix}, & \sup\limits_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=m-1}} \inf\limits_{\substack{x\in\mathfrak{D}_1,\, x\neq 0\\ x\perp\mathcal{L}}} \sup\limits_{\substack{y\in\mathfrak{D}_2\\ y\neq 0}} \lambda_+\begin{pmatrix}x\\y\end{pmatrix}, \\ \inf\limits_{\substack{\mathcal{L}\subset\mathfrak{D}_1\\ \dim\mathcal{L}=m}} \max\limits_{\substack{x\in\mathcal{L}\\ x\neq 0}} \sup\limits_{y\in\mathfrak{D}_2} \dfrac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2}, & \sup\limits_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=m-1}} \inf\limits_{\substack{x\in\mathfrak{D}_1,\, x\neq 0\\ x\perp\mathcal{L}}} \sup\limits_{y\in\mathfrak{D}_2} \dfrac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2}, \end{array} \tag{2.10.9}$$

*then*

$$\mu_m = \begin{cases} d & \textit{if} \;\; m = 1, 2, \dots, \kappa, \\ \lambda_{\rm e} & \textit{if} \;\; m \geq \kappa + N + 1. \end{cases} \tag{2.10.10}$$

*In the particular case* $\max\sigma(D) = d < a = \min\sigma(A)$, *we have* $\kappa = 0$.

Obviously, in (2.10.7) the conditions $x \neq 0$ and $y \neq 0$ can be replaced by $\|x\| = 1$ and $\|y\| = 1$, respectively, and in (2.10.8) the variation over $y$ can be restricted to vectors with $\big\|(x\ y)^{\rm t}\big\| = 1$; the same applies to (2.10.9).

**Remark 2.10.5** In all variational principles above, the dimension restriction concerns only the first component. This was already observed in [DES00a] and [GS99], where the variational characterization (2.10.8) using the Rayleigh functional $\mathfrak{A}$ was first derived under different assumptions and without index shift, *i.e.* with $\kappa = 0$.

To prove Theorem 2.10.4, we relate the functional $\lambda_+$ to the Rayleigh functional $\mathfrak{A}$ of the block operator matrix $\mathcal{A}$ and to a generalized Rayleigh functional of the form $\mathfrak{s}_1$ defining the first Schur complement.

**Lemma 2.10.6** *Let* $x \in \mathfrak{D}_1$, $x \neq 0$. *Then*

i) *for* $\lambda \in \rho(D)$ *and* $y := -(D-\lambda)^{-1}B^*x$ *we have*
$$\frac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2} = \lambda + \frac{\mathfrak{s}_1(\lambda)[x]}{\big\|\binom{x}{y}\big\|^2};$$

ii) *the function* $\mathfrak{s}_1(\cdot)[x]$ *has at most one zero* $p(x) \in (d, \infty)$;

iii) *if* $\mu \in (d, \infty)$ *is such that* $\mathfrak{s}_1(\mu)[x] \leq 0$, *then*
$$\frac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2} \leq \lambda_+\begin{pmatrix}x\\y\end{pmatrix} \leq \mu, \quad y \in \mathfrak{D}_2,\, y \neq 0;$$

iv) *if a zero* $p(x)$ *of* $\mathfrak{s}_1(\cdot)[x]$ *exists, it satisfies*
$$p(x) = \max_{\substack{y\in\mathfrak{D}_2\\ y\neq 0}} \lambda_+\begin{pmatrix}x\\y\end{pmatrix} = \max_{y\in\mathfrak{D}_2} \frac{\mathfrak{A}\big[\binom{x}{y}\big]}{\big\|\binom{x}{y}\big\|^2}. \tag{2.10.11}$$

***Proof.*** Let $x \in \mathfrak{D}_1$, $x \neq 0$, be fixed.

i) By definition of $\mathfrak{A}$ and $\mathfrak{a}$, we have, for $\lambda \in \rho(D)$, $y := -(D-\lambda)^{-1}B^*x$,

$$\begin{aligned}\mathfrak{A}\left[\begin{pmatrix}x\\y\end{pmatrix}\right]-\lambda\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^2 &= (\mathfrak{a}-\lambda)[x] - \big(B^*x,(D-\lambda)^{-1}B^*x\big)\\ &\quad - \big((D-\lambda)^{-1}B^*x,B^*x\big)+\big(B^*x,(D-\lambda)^{-1}B^*x\big)\\ &= \mathfrak{s}_1(\lambda)[x].\end{aligned}$$

ii) By the definition of $\mathfrak{s}_1$ in Proposition 2.10.1 i), we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathfrak{s}_1(\lambda)[x] = -\|x\|^2 - \big((D-\lambda)^{-2}B^*x,B^*x\big) \leq -\|x\|^2, \quad \lambda \in (d,\infty).$$

This shows that $\mathfrak{s}_1(\cdot)[x]$ is a strictly decreasing function on $(d,\infty)$ and $\lim_{\lambda\to\infty}\mathfrak{s}_1(\lambda)[x] = -\infty$, which proves ii).

iii) Let $y \in \mathfrak{D}_2$, $y \neq 0$, and define the $2\times 2$ matrix

$$\mathcal{A}_{x,y} := \begin{pmatrix} \dfrac{\mathfrak{a}[x]}{\|x\|^2} & \dfrac{(y,B^*x)}{\|x\|\,\|y\|} \\[2ex] \dfrac{(B^*x,y)}{\|x\|\,\|y\|} & \dfrac{\mathfrak{d}[y]}{\|y\|^2}\end{pmatrix} \in M_2(\mathbb{C}).$$

It is easy to see that

$$\frac{\mathfrak{A}\left[\binom{x}{y}\right]}{\left\|\binom{x}{y}\right\|^2} = \frac{\mathfrak{a}[x]+(B^*x,y)+(y,B^*x)+\mathfrak{d}[y]}{\|x\|^2+\|y\|^2} = \frac{\Big(\mathcal{A}_{x,y}\binom{\|x\|}{\|y\|},\binom{\|x\|}{\|y\|}\Big)_{\mathbb{C}^2}}{\|x\|^2+\|y\|^2} \leq \lambda_+\begin{pmatrix}x\\y\end{pmatrix}$$

since, by definition, $\lambda_+(x,y)$ is the largest of the two eigenvalues of the matrix $\mathcal{A}_{x,y}$ and hence the maximum of its numerical range. To prove the second inequality in ii), let first $y \in \mathcal{D}(D)$, $y \neq 0$. Since $\lambda_+ := \lambda_+(x,y)$ is an eigenvalue of $\mathcal{A}_{x,y}$, we have $\det(\mathcal{A}_{x,y}-\lambda_+) = 0$ or, equivalently,

$$\big(\mathfrak{a}-\lambda_+\big)[x]\,\big((D-\lambda_+)y,y\big) = \big|(B^*x,y)\big|^2. \tag{2.10.12}$$

If $\lambda_+ \leq d$, then the assertion is obvious. If $\lambda_+ > d = \max\sigma(D)$, then the right hand side of (2.10.12) can be estimated by the Cauchy–Schwarz inequality with respect to the scalar product $\big((\lambda_+-D)^{-1}\cdot,\cdot\big)$:

$$\begin{aligned}\big|(B^*x,y)\big|^2 &= \Big|\big((\lambda_+-D)^{-1}B^*x,(\lambda_+-D)y\big)\Big|^2\\ &\leq \big((\lambda_+-D)^{-1}B^*x,B^*x\big)\,\big((\lambda_+-D)^{-1}(\lambda_+-D)y,(\lambda_+-D)y\big)\\ &= \big((\lambda_+-D)^{-1}B^*x,B^*x\big)\,\big((\lambda_+-D)y,y\big).\end{aligned} \tag{2.10.13}$$

Hence, by (2.10.12) and (2.10.13),

$$-\big(\mathfrak{a}-\lambda_+\big)[x] \leq \big((\lambda_+-D)^{-1}B^*x,B^*x\big),$$

which is equivalent to $\mathfrak{s}_1(\lambda_+)[x] \geq 0$. On the other hand, $\mathfrak{s}_1(\mu)[x] \leq 0$ and we have shown in the proof of ii) that $\mathfrak{s}_1(\cdot)[x]$ is strictly decreasing on $(d, \infty)$. Thus $\lambda_+(x,y) = \lambda_+ \leq \mu$. Since $\lambda_+(x,y)$ is continuous in $y$ with respect to the graph norm of $D$, this inequality continues to hold for $y \in \mathfrak{D}_2$.

iv) The inequalities "$\geq$" for $p(x)$ in (2.10.11), with sup instead of max, follow from iii) with $\mu = p(x)$; note that, for $y = 0$, we have $\mathfrak{A}[\binom{x}{y}]/\|\binom{x}{y}\|^2 = \mathfrak{a}[x]/\|x\|^2 \leq p(x)$ since $0 = \mathfrak{s}_1\big(p(x)\big)[x] \geq \mathfrak{a}[x] - p(x)\|x\|^2$. If $B^*x = 0$, then $\mathfrak{s}_1(\lambda)[x] = \mathfrak{a}[x] - \lambda\|x\|^2$ and hence, for every $y \in \mathfrak{D}_2$, $y \neq 0$,

$$p(x) = \frac{\mathfrak{a}[x]}{\|x\|^2} = \lambda_+\begin{pmatrix} x \\ y \end{pmatrix}.$$

If $B^*x \neq 0$, then i) with $\lambda = p(x)$ and iii) yield that

$$p(x) = \frac{\mathfrak{A}\Big[\Big(\begin{smallmatrix} x \\ -(D-p(x))^{-1}B^*x \end{smallmatrix}\Big)\Big]}{\Big\|\Big(\begin{smallmatrix} x \\ -(D-p(x))^{-1}B^*x \end{smallmatrix}\Big)\Big\|^2} \leq \lambda_+\begin{pmatrix} x \\ -(D-p(x))^{-1}B^*x \end{pmatrix}. \qquad \square$$

***Proof of Theorem* 2.10.4.** For $x \in \mathfrak{D}_1$, $x \neq 0$, let $p(x)$ be the zero of $\mathfrak{s}_1(\cdot)[x]$ according to Lemma 2.10.6 ii); if no zero exists, set $p(x) := -\infty$.

The facts that $S_1$ is a holomorphic operator function of type (B), that $\mathfrak{s}_1(\cdot)[x]$ is decreasing for $x \in \mathfrak{D}_1$, and that there exists a $\gamma \in (d, \infty)$ with $\kappa_-(\gamma) < \infty$ imply that all assumptions of [EL04, Theorem 2.1] are satisfied for the operator function $S_1$ on the interval $(d, \infty)$ (see also [EL04, Proposition 2.13]). Now Theorem 2.1 in [EL04] implies that there exists an interval $(d, \alpha) \subset \rho\big(\overline{\mathcal{A}}\big)$, that $\lambda_{\mathrm{e}} > d$ (see [EL04, Lemma 2.9]), and that, with the generalized Rayleigh functional $p$ of $S_1$ defined as above,

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathfrak{D}_1 \\ \dim \mathcal{L} = \kappa + n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} p(x) = \max_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = \kappa + n - 1}} \inf_{\substack{x \in \mathfrak{D}_1,\, x \neq 0 \\ x \perp \mathcal{L}}} p(x)$$

for $n = 1, 2, \ldots, N$ and

$$\inf_{\substack{\mathcal{L} \subset \mathfrak{D}_1 \\ \dim \mathcal{L} = m}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} p(x) = \sup_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = m - 1}} \inf_{\substack{x \in \mathfrak{D}_1,\, x \neq 0 \\ x \perp \mathcal{L}}} p(x) = \lambda_{\mathrm{e}}$$

for $m \geq \kappa + N + 1$. Now the relations (2.10.7) and (2.10.8) and the second line in (2.10.10) follow from (2.10.11) in Lemma 2.10.6 iv).

It remains to be proved that $\mu_m = d$ for $m \leq \kappa$. Let $\mu_m$ denote either the first or the third expression in (2.10.9); the other two cases are similar. For arbitrary $\mu \in (d, \alpha)$, we have $\kappa_-(\mu) = \kappa \geq m$ and hence there exists a subspace $\mathcal{L}_\mu \subset \mathcal{L}_{(-\infty,0)}\big(S(\mu)\big)$ with $\dim \mathcal{L}_\mu = m$. For $x \in \mathcal{L}_\mu$, $x \neq 0$, we have $\mathfrak{s}_1(\mu)[x] \leq 0$, and hence $\mathfrak{A}[\binom{x}{y}]/\|\binom{x}{y}\|^2 \leq \lambda_+\binom{x}{y} \leq \mu$ for all $y \in \mathfrak{D}_2$, $y \neq 0$, by Lemma 2.10.6 iii). Since $\mu \in (d, \infty)$ was arbitrary, this proves $\mu_m \leq d$. In order to show that $\mu_m \geq d$, let $x \in \mathfrak{D}_1$, $x \neq 0$, be arbitrary. Then, for every $\varepsilon > 0$, we can choose $y_\varepsilon \in \mathfrak{D}_2$, $\|y_\varepsilon\| = 1$, so that $\mathfrak{d}[y_\varepsilon] > d - \varepsilon$. According to (2.10.5), we have $\lambda_+(x, y_\varepsilon) \geq \mathfrak{d}[y_\varepsilon] > d - \varepsilon$ and, since $\lim_{t\to\infty} \mathfrak{A}[\binom{x}{ty_\varepsilon}]/\|\binom{x}{ty_\varepsilon}\|^2 = \mathfrak{d}[y_\varepsilon]$, also $\mathfrak{A}[\binom{x}{ty_\varepsilon}]/\|\binom{x}{ty_\varepsilon}\|^2 > d - \varepsilon$ for $t \in \mathbb{R}$ large enough. Hence $\mu_m \geq d - \varepsilon$.

The last claim is obvious since $d < a$ implies that $(d, a) \subset \rho\big(\overline{\mathcal{A}}\big)$ (see Theorem 2.5.18) and that $S_1(\gamma)$ is uniformly positive for $\gamma \in (d, a)$. □

In general, it is not easy to check that the assumption $\kappa_-(\gamma) < \infty$ holds for some $\gamma \in \mathbb{R}$ and to calculate the index shift $\kappa$. Some information may be obtained from the numbers $\mu_m$ defined in Theorem 2.10.4.

**Remark 2.10.7** Let $\mu_m$, $m = 1, 2, \ldots$, be as in Theorem 2.10.4.

i) If $\mu_m > d$ for some $m \in \mathbb{N}$, then $\kappa_-(\gamma) < \infty$ for some $\gamma \in (d, \infty)$.
ii) If $\kappa_-(\gamma) < \infty$ for some $\gamma \in (d, \infty)$ and $\dim \mathcal{H}_1 > \kappa$, then $\mu_m > d$ for some $m \in \mathbb{N}$.

In both cases, the index shift $\kappa := \kappa_-(\alpha)$ in Theorem 2.10.4 with $\alpha \in (d, \infty)$ such that $(d, \alpha) \subset \rho\big(\overline{\mathcal{A}}\big)$ is given by

$$\kappa = \max\big\{m \in \mathbb{N} : \mu_m = d\big\}. \tag{2.10.14}$$

***Proof.*** Let $\mu_m$ be the first expression in (2.10.9); the other three cases are similar. Formula (2.10.14) is obvious from (2.10.10).

i) Assume that $m \in \mathbb{N}$ is such that $\mu_m > d$. The claim is obvious if $\dim \mathcal{H}_1 < \infty$. Otherwise, suppose that $\kappa_-(\mu) = \infty$ for all $\mu \in (d, \infty)$. Then, for every $\mu \in (d, \infty)$, there exists a subspace $\mathcal{L}_\mu \subset \mathcal{L}_{(-\infty,0)}\big(S(\mu)\big)$ with $\dim \mathcal{L}_\mu = m$. From Lemma 2.10.3 iii) it follows that $\lambda_+(x, y) \leq \mu$ for all $x \in \mathcal{L}_\mu$, $y \in \mathfrak{D}_2$, $x, y \neq 0$. This implies that $\mu_m \leq \mu$ for all $\mu \in (d, \infty)$ and hence $\mu_m \leq d$, a contradiction.

ii) The claim is immediate from (2.10.7) and (2.10.8). □

In the second part of this section, we prove variational principles for $\mathcal{J}$-self-adjoint diagonally dominant block operator matrices. First we consider the case that the quadratic numerical range, and hence the spectrum, is real (see [LLT02, Theorem 4.2] for the case of bounded $B$).

**Theorem 2.10.8** *Suppose that $A = A^*$ is bounded from below, $D = D^*$ is bounded from above, $\max \sigma(D) =: d < a := \min \sigma(A)$, $B$ is closed, $C = -B^*$, and $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$, $\mathcal{D}(|D|^{1/2}) \subset \mathcal{D}(B)$* (i.e. *$\mathcal{A}$ satisfies condition* (a) *in Proposition* 2.10.1 *and is $\mathcal{J}$-self-adjoint*). *Assume that $W^2(\mathcal{A}) \subset \mathbb{R}$ and that there is a $\gamma \in \mathbb{R}$ with $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$. Define*

$$\lambda_{\mathrm{e}} := \min\big(\sigma_{\mathrm{ess}}(\mathcal{A}) \cap (\gamma, \infty)\big), \quad \lambda_{\mathrm{e}}' := \max\big(\sigma_{\mathrm{ess}}(\mathcal{A}) \cup \{\gamma\}\big). \tag{2.10.15}$$

*If* $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, *is the finite or infinite sequence of eigenvalues of* $\mathcal{A}$ *in* $(\gamma, \lambda_e)$ *counted with multiplicities, then*

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=n}} \max_{\substack{x\in\mathcal{L}\\ x\neq 0}} \min_{\substack{y\in\mathcal{D}(D)\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \quad n = 1, 2, \dots, N. \tag{2.10.16}$$

*If* $A$ *is bounded and* $\lambda'_1 \ge \lambda'_2 \ge \dots \ge \lambda'_{N'}$, $N' \in \mathbb{N}_0 \cup \{\infty\}$, *is the finite or infinite sequence of eigenvalues of* $\mathcal{A}$ *in* $(\lambda'_e, \infty)$ *counted with multiplicities, then*

$$\lambda'_n = \max_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=n}} \min_{\substack{x\in\mathcal{L}\\ x\neq 0}} \min_{\substack{y\in\mathcal{D}(D)\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \quad n = 1, 2, \dots, N'. \tag{2.10.17}$$

*For* $n > N$, *the right hand side of* (2.10.16) *equals* $\lambda_e$*; for* $n > N'$, *the right hand side of* (2.10.17) *equals* $\lambda'_e$.

Like in the self-adjoint case, the proof of Theorem 2.10.8 relies on the variational principle for unbounded operator functions from [EL04]. In contrast to the self-adjoint case (see Lemma 2.10.6), the function $(S_1(\cdot)x, x)$ is no longer monotone and hence does not readily furnish a unique zero defining the generalized Rayleigh functional $p(x)$ required in [EL04]. However, we can use the monotonicity of the derivative.

**Lemma 2.10.9** *Let* $\gamma \in \mathbb{R}$ *be such that* $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$ *and let* $x \in \mathcal{D}(A)$, $x \neq 0$. *Then*

i) *the function* $\big(S_1(\cdot)x, x\big)$ *has exactly one zero* $p(x) \in (\gamma, \infty)$ *and*

$$\big(S_1(\lambda)x, x\big) \begin{cases} > 0, & \lambda \in \big(\gamma, p(x)\big),\\ < 0, & \lambda \in \big(p(x), \infty\big);\end{cases}$$

ii) *if* $\mu \in (\gamma, \infty)$ *is such that* $\big(S_1(\mu)x, x\big) \ge 0$, *then*

$$\lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \ge \mu, \quad y \in \mathcal{D}(D),\ y \neq 0;$$

iii) *the zero* $p(x)$ *satisfies*

$$\min_{\substack{y\in\mathcal{D}(D)\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} = p(x). \tag{2.10.18}$$

***Proof.*** Let $x \in \mathcal{D}(A)$, $x \neq 0$, be fixed.

i) Differentiating the formula $S_1(\lambda) = A - \lambda + B(D-\lambda)^{-1}B^*$ twice, we find that

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\big(S_1(\lambda)x, x\big) = \big((D-\lambda)^{-3}B^*x, B^*x\big) \le 0, \quad \lambda \in (d, \infty);$$

thus $\big(S_1(\cdot)x, x\big)$ is concave on $(d, \infty)$. Moreover, $\big(S_1(\lambda)x, x\big) \to -\infty$ for $\lambda \to \infty$. Hence the function $\big(S_1(\cdot)x, x\big)$ has at most two zeroes in $(d, \infty)$.

By Proposition 2.6.10, the assumption $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$ implies that $S_1(\gamma)$ is uniformly positive. Therefore, the function $\big(S_1(\cdot)x, x\big)$ has exactly one zero in $(\gamma, \infty)$ with the desired property.

ii) Let $\lambda_+ := \lambda_+(x, y)$. Since $\gamma < \overline{\Lambda_+(\mathcal{A})}$, we have $\lambda_+ > \gamma > d$. By a similar computation as in the proof of Lemma 2.10.6 iii) (note that now we have a minus sign on the right hand side of (2.10.12)), we obtain $\big(S_1(\lambda_+)x, x\big) \le 0$. On the other hand, $\big(S_1(\mu)x, x\big) \ge 0$ by assumption. Hence, by i), we obtain $\mu \le p(x) \le \lambda_+$.

iii) The inequality "$\ge$" follows from ii) with $\mu = p(x)$. If $B^*x = 0$, then $\big(S_1(\lambda)x, x\big) = \big((A-\lambda)x, x\big)$ and hence, for every $y \in \mathcal{D}(D)$, $y \neq 0$,

$$p(x) = \frac{(Ax, x)}{\|x\|^2} = \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}.$$

If $B^*x \neq 0$, then (2.5.3) in Lemma 2.5.7 i) with $C \subset -B^*$ shows that $p(x)$ is one of the two zeroes of $\Delta\big(x, -(D-\cdot)^{-1}B^*x; \cdot\big)$. Since $\lambda_\pm(x, y)$ are the two zeroes of the quadratic polynomial $\Delta(x, y; \cdot)$ and $p(x) \ge \gamma > \overline{\Lambda_-(\mathcal{A})}$, we conclude that

$$p(x) = \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}, \quad y := -\big(D - p(x)\big)^{-1} B^*x \in \mathcal{D}(D). \qquad \square$$

***Proof of Theorem* 2.10.8.** By Proposition 2.6.8, we have the inclusion $\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})} \subset \mathbb{R}$. As in the proof of Theorem 2.10.4, we use Lemma 2.10.9 and the variational principle from [EL04] for $S_1$ to derive formulae (2.10.16), (2.10.17). Note that for applying [EL04, Theorem 2.1], $\big(S_1(\cdot)x, x\big)$ need not be monotone, but a sign change at $p(x)$ suffices. $\square$

For bounded $\mathcal{J}$-self-adjoint block operator matrices, Theorem 2.10.8 has been generalized to the case that the quadratic numerical range is no longer real (see [LLMT05, Theorem 5.3]). As in the self-adjoint case with overlapping spectra of the diagonal elements, this may lead to an index shift in the variational principle.

**Theorem 2.10.10** *Let $\mathcal{A}$ be bounded and $\mathcal{J}$-self-adjoint. Let the number $\mu \in \mathbb{R}$ be given as in* (1.10.2) *and assume that there is a $\lambda_0 \in (\mu, \infty)$ such that $\dim \mathcal{L}_{(-\infty,0)}\big(S_1(\lambda_0)\big) < \infty$. Then there exists an $\alpha > \mu$ with $(\mu, \alpha) \subset \rho(\mathcal{A})$. Define*

$$\kappa := \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) < \infty$$

*and*

$$\lambda_{\mathrm{e}} := \min\big(\sigma_{\mathrm{ess}}(\mathcal{A}) \cap (\alpha, \infty)\big), \quad \lambda_{\mathrm{e}}' := \max\big(\sigma_{\mathrm{ess}}(\mathcal{A}) \cup \{\alpha\}\big).$$

*If* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, *is the finite or infinite sequence of eigenvalues of* $\mathcal{A}$ *in* $(\alpha, \lambda_{\mathrm{e}})$ *counted with multiplicities, then*

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=\kappa+n}} \max_{\substack{x\in\mathcal{L}\\ x\neq 0}} \min_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \quad n = 1, 2, \ldots, N. \tag{2.10.19}$$

*If* $\lambda_1' \ge \lambda_2' \ge \cdots \ge \lambda_{N'}'$, $N' \in \mathbb{N}_0 \cup \{\infty\}$, *is the finite or infinite sequence of eigenvalues of* $\mathcal{A}$ *in* $(\lambda_{\mathrm{e}}', \infty)$ *counted with multiplicities, then*

$$\lambda_n' = \max_{\substack{\mathcal{L}\subset\mathcal{H}_1\\ \dim\mathcal{L}=n}} \min_{\substack{x\in\mathcal{L}\\ x\neq 0}} \min_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \quad n = 1, 2, \ldots, N'. \tag{2.10.20}$$

*For* $n > N$, *the right hand side of* (2.10.19) *equals* $\lambda_{\mathrm{e}}$; *for* $n > N'$, *the right hand side of* (2.10.20) *equals* $\lambda_{\mathrm{e}}'$.

***Proof.*** The proof of this theorem is similar to the proof of Theorem 2.10.8. Note that here it can only be shown that for $x \in \mathcal{H}_1$, $x \neq 0$, the function $\big(S_1(\cdot)x, x\big)$ has at most one zero in $(\mu, \infty)$, not exactly one; the Rayleigh functional $p(x)$ is defined as this zero if it exists, and $-\infty$ otherwise. There is no index shift in (2.10.20) since $S_1(\lambda) \le 0$ for $\lambda$ large enough. For details we refer to the proof in [LLMT05, Section 5]. □

**Remark 2.10.11** For a bounded block operator matrix $\mathcal{A}$, Theorem 2.10.8 is a special case of Theorem 2.10.10; in fact, by Proposition 2.6.10, the condition $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$ of Theorem 2.10.8 implies that $S_1(\gamma)$ is uniformly positive and hence $\kappa = 0$.

## 2.11 Eigenvalue estimates

The variational principles established in the previous section for (essentially) self-adjoint and $\mathcal{J}$-self-adjoint block operator matrices are now used to derive upper and lower bounds for eigenvalues. The bounds are given in terms of the eigenvalues of the diagonal elements $A$, $D$ in the diagonally dominant and upper dominant case and in terms of the eigenvalues of the operators $BB^*$ and $B^*B$ in the off-diagonally dominant case. Again we formulate and prove all results for eigenvalues to the right of $d = \max\sigma(D)$; analogous results hold for eigenvalues to the left of $a = \min\sigma(A)$.

First we give a simple eigenvalue estimate from below for the diagonally dominant, upper dominant, and off-diagonally dominant self-adjoint case.

**Proposition 2.11.1** *Suppose that the* (*essentially*) *self-adjoint block operator matrix* $\mathcal{A}$ *satisfies the assumptions of Theorem* 2.10.4, *define* $\lambda_{\mathrm{e}}$

*and $\kappa$ as therein, and let $d := \max\sigma(D)$. Let $\nu_1(A) \le \nu_2(A) \le \cdots \le \nu_M(A)$, $M \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of the diagonal element $A$ below $\sigma_{\rm ess}(A)$ counted with multiplicities, and set $\nu_k(A) := \min\sigma_{\rm ess}(A)$ for $k > M$. Then $\nu_k(A) \le d$, $k = 1, 2, \dots, \kappa$, and the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, of $\overline{\mathcal{A}}$ in $(d, \lambda_{\rm e})$ satisfy the estimate*

$$\lambda_n \ge \nu_{\kappa+n}(A), \quad n = 1, 2, \dots, \min\{N, M - \kappa\}. \tag{2.11.1}$$

***Proof.*** The estimates are immediate consequences of Theorem 2.10.4: they follow from (2.10.7), from the inequality $(Ax, x)/\|x\|^2 \le \lambda_+(x, y)$, and from the standard variational principle for the self-adjoint operator $A$, which implies that $\nu_k(A) \le \mu_k$ for $k = 1, 2, \dots, M$. □

The next theorem provides more subtle upper bounds for the eigenvalues of diagonally dominant and upper dominant (essentially) self-adjoint block operator matrices and, if the diagonal entry $D$ is bounded, also lower bounds refining (2.11.1).

**Theorem 2.11.2** *Let $A = A^*$ be bounded from below, $D = D^*$ bounded from above, $d := \max\sigma(D)$, $B$ closed, and $C = B^*$. Assume that the block operator matrix $\mathcal{A}$ fulfils condition* (a) *or* (b) *in Proposition* 2.10.1 *and that there is a $\gamma \in (d, \infty)$ with $\kappa_-(\gamma) < \infty$. Define $\lambda_{\rm e}$ and $\kappa$ as in Theorem* 2.10.4. *Let $\nu_1(A) \le \nu_2(A) \le \cdots \le \nu_M(A)$, $M \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of the diagonal element $A$ below $\sigma_{\rm ess}(A)$ counted with multiplicities, set $\nu_k(A) := \min\sigma_{\rm ess}(A)$ for $k > M$, and let $a', b' \ge 0$, $a'' \in \mathbb{R}$, $b'' \ge 0$ be so that*

$$\|B^*x\|^2 \le a'\,\|x\|^2 + b'\,\mathfrak{a}[x], \quad x \in \mathcal{D}(|A|^{1/2}), \tag{2.11.2}$$

$$\|B^*x\|^2 \ge a''\|x\|^2 + b''\mathfrak{a}[x], \quad x \in \mathcal{D}(|A|^{1/2}). \tag{2.11.3}$$

*If $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, are the eigenvalues of $\overline{\mathcal{A}}$ in $(d, \lambda_{\rm e})$ counted with multiplicities, then*

$$\lambda_n \le \frac{\nu_{\kappa+n}(A) + \max\sigma(D)}{2} + \sqrt{\left(\frac{\nu_{\kappa+n}(A) - \max\sigma(D)}{2}\right)^2 + b'\nu_{\kappa+n}(A) + a'}$$

*for $n = 1, \dots, N$, and if, in addition, $D$ is bounded, then*

$$\lambda_n \ge \frac{\nu_{\kappa+n}(A) + \min\sigma(D)}{2} + \sqrt{\left(\frac{\nu_{\kappa+n}(A) - \min\sigma(D)}{2}\right)^2 + \big(b''\nu_{\kappa+n}(A) + a''\big)_+}$$

*for $n = 1, \dots, N$, where $(t)_+ := \max\{t, 0\}$ for $t \in \mathbb{R}$. In the particular case $\max\sigma(D) = d < a = \min\sigma(A)$, we have $\kappa = 0$.*

**Remark 2.11.3** i) Note that the inequality (2.11.3) always holds with $a'' = \min\sigma(BB^*)$, $b'' = 0$; if $B$ is bounded, then (2.11.2) always holds with

$a' = \|B\|$, $b' = 0$. Thus Theorem 2.11.2 generalizes [LLT02, Theorem 5.1] where the diagonally dominant case with bounded $B$ was considered.

ii) If $A$ has compact resolvent, then $\sigma_{\rm ess}(\mathcal{A}) \cap (d,\infty) = \emptyset$ and for every $\gamma \in (d,\infty)$ we have $\kappa_-(\gamma) < \infty$. This follows from [EL04, Theorem 4.5].

***Proof of Theorem* 2.11.2.** By assumption (a) or (b) in Proposition 2.10.1, we have $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$. Hence $B^*$ is $|A|^{1/2}$-bounded and thus $a', b' \geq 0$ with (2.11.2) exist (see Remark 2.1.3 ii)). The existence of $a'' \geq 0$, $b'' \geq 0$ with (2.11.3) is clear (see *e.g.* Remark 2.11.3); note that $a''$ may also be chosen to be negative.

The proof relies on the fact that, for $\beta \geq 0$, the function

$$f(s,t) := s + t + \sqrt{(s-t)^2 + \beta}, \quad s,t \in \mathbb{R}, \tag{2.11.4}$$

is increasing in $s$ and $t$. In order to obtain the upper bound for $\lambda_n$, let $\varepsilon > 0$. There exists a subspace $\mathcal{L}_\varepsilon \subset \mathcal{L}_{(-\infty,\nu_{\kappa+n}(A)+\varepsilon]}(A)$ with $\dim \mathcal{L}_\varepsilon = \kappa + n$. By Theorem 2.10.4, we have

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=\kappa+n}} \max_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \max_{\substack{y\in\mathcal{D}(D)\\ \|y\|=1}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \leq \max_{\substack{x\in\mathcal{L}_\varepsilon\\ \|x\|=1}} \max_{\substack{y\in\mathcal{D}(D)\\ \|y\|=1}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}.$$

Using first the monotonicity of $f$ with the estimates $(Ax,x) \leq \nu_{\kappa+n}(A) + \varepsilon$, $(Dy,y) \leq \max\sigma(D) = d$, and then inequality (2.11.2), we obtain that, for all $x \in \mathcal{L}_\varepsilon$, $y \in \mathcal{D}(D)$, $\|x\| = \|y\| = 1$,

$$\begin{aligned} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} &= \frac{(Ax,x)+(Dy,y)}{2} + \sqrt{\left(\frac{(Ax,x)-(Dy,y)}{2}\right)^2 + \big|(By,x)\big|^2} \\ &\leq \frac{\nu_{\kappa+n}(A)+\varepsilon+d}{2} + \sqrt{\left(\frac{\nu_{\kappa+n}(A)+\varepsilon-d}{2}\right)^2 + b'\nu_{\kappa+n}(A) + a'}\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the upper estimate for $\lambda_n$ follows. Note that, in fact, the analysis with $\varepsilon$ is only necessary if $\nu_{\kappa+n}(A) = \min\sigma_{\rm ess}(A)$.

To prove the lower estimate for $\lambda_n$, let $\mathcal{L} \subset \mathcal{D}(A)$ with $\dim\mathcal{L} = \kappa + n$ be an arbitrary subspace. By the classical variational principle for the semi-bounded operator $A$, there exists an $x_\mathcal{L} \in \mathcal{L}$, $\|x_\mathcal{L}\| = 1$, such that $(Ax_\mathcal{L}, x_\mathcal{L}) \geq \nu_{\kappa+n}(A)$. Again by Theorem 2.10.4, we have

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=\kappa+n}} \max_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \max_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \geq \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=\kappa+n}} \lambda_+\begin{pmatrix} x_\mathcal{L}\\ B^*x_\mathcal{L}\end{pmatrix}.$$

If we use the monotonicity of $f$, set $d_- := \min\sigma(D)$, and note that (2.11.3) implies $\|B^*x\|^2 \geq \big(a''\|x\|^2 + b''\mathfrak{a}[x]\big)_+$, $x \in \mathcal{D}(|A|^{1/2})$, we arrive at

$$\begin{aligned}
&\lambda_+\begin{pmatrix} x_{\mathcal{L}} \\ B^* x_{\mathcal{L}} \end{pmatrix} \\
&= \frac{(Ax_{\mathcal{L}}, x_{\mathcal{L}})}{2} + \frac{(DB^*x_{\mathcal{L}}, B^*x_{\mathcal{L}})}{2\,\|B^*x_{\mathcal{L}}\|^2} + \sqrt{\left(\frac{(Ax_{\mathcal{L}}, x_{\mathcal{L}})}{2} - \frac{(DB^*x_{\mathcal{L}}, B^*x_{\mathcal{L}})}{2\,\|B^*x_{\mathcal{L}}\|^2}\right)^2 + \|B^*x_{\mathcal{L}}\|^2} \\
&\geq \frac{\nu_{\kappa+n}(A) + d_-}{2} + \sqrt{\left(\frac{\nu_{\kappa+n}(A) - d_-}{2}\right)^2 + \big(b''\nu_{\kappa+n}(A) + a''\big)_+}\,. \qquad \square
\end{aligned}$$

As a consequence of Theorem 2.11.2, we obtain asymptotic estimates for the eigenvalues of the block operator matrix $\mathcal{A}$ if $\nu_k(A) \to \infty$ for $k \to \infty$.

**Corollary 2.11.4** *If, under the assumptions of Theorem* 2.11.2*, we have* $\nu_k(A) \to \infty$ *for* $k \to \infty$*, then, for* $n \to \infty$*,*

$$\lambda_n \leq \nu_{\kappa+n}(A) + b' + \frac{b' \max\sigma(D) + a' - b'^2}{\nu_{\kappa+n}(A) - \max\sigma(D)} + \mathrm{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right), \quad (2.11.5)$$

$$\lambda_n \geq \nu_{\kappa+n}(A) + b'' + \frac{b'' \min\sigma(D) + a'' - b''^2}{\nu_{\kappa+n}(A) - \min\sigma(D)} + \mathrm{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right). \quad (2.11.6)$$

***Proof.*** The upper estimate in Theorem 2.11.2 with $d = \max\sigma(D)$ yields

$$\begin{aligned}
\lambda_n &\leq \frac{\nu_{\kappa+n}(A) + d}{2} + \frac{\nu_{\kappa+n}(A) - d}{2}\sqrt{1 + \big(b'\nu_{\kappa+n}(A) + a'\big)\left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^{-2}} \\
&= \frac{\nu_{\kappa+n}(A) + d}{2} + \frac{\nu_{\kappa+n}(A) - d}{2}\left[1 + \frac{1}{2}\big(b'\nu_{\kappa+n}(A) + a'\big)\left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^{-2}\right. \\
&\qquad \left. - \frac{1}{8}\big(b'\nu_{\kappa+n}(A) + a'\big)^2\left(\frac{\nu_{\kappa+n}(A) - d}{2}\right)^{-4} + \mathrm{O}\left(\frac{1}{\nu_{\kappa+n}(A)^3}\right)\right] \\
&= \nu_{\kappa+n}(A) + \frac{b'\nu_{\kappa+n}(A) + a'}{\nu_{\kappa+n}(A) - d} - \frac{(b'\nu_{\kappa+n}(A) + a')^2}{(\nu_{\kappa+n}(A) - d)^3} + \mathrm{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right) \\
&= \nu_{\kappa+n}(A) + b' + \frac{b'd + a' - b'^2}{\nu_{\kappa+n}(A) - d} + \mathrm{O}\left(\frac{1}{\nu_{\kappa+n}(A)^2}\right).
\end{aligned}$$

This proves (2.11.5); the proof of (2.11.6) is similar if we use $(t)_+ \geq t$. $\square$

Next we derive upper and lower bounds for eigenvalues of off-diagonally dominant self-adjoint block operator matrices in terms of eigenvalues of the operator $BB^*$; as in Section 2.10, we restrict ourselves to bounded $A$ and $D$.

**Theorem 2.11.5** *Let* $A = A^*$ *and* $D = D^*$ *be bounded,* $d := \max\sigma(D)$*,* $B$ *closed, and* $C = B^*$ *(so that* $\mathcal{A}$ *is self-adjoint). Suppose that there exists a* $\gamma \in (d, \infty)$ *with* $\kappa_-(\gamma) < \infty$*. Define* $\lambda_{\mathrm{e}}$ *and* $\kappa$ *as in Theorem* 2.10.4*. Let* $\nu_1(BB^*) \leq \nu_2(BB^*) \leq \ldots \leq \nu_M(BB^*)$*,* $M \in \mathbb{N}_0 \cup \{\infty\}$*, be the eigenvalues* *of* $BB^*$ *below* $\sigma_{\rm ess}\big(BB^*\big)$ *counted with multiplicities and set* $\nu_k(BB^*) := \min\sigma_{\rm ess}\big(BB^*\big)$ *for* $k > M$. *If* $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, *are the eigenvalues of* $\mathcal{A}$ *in* $(d, \lambda_{\rm e})$ *counted with multiplicities, then, for* $n = 1, \ldots, N$,

$$\lambda_n \ge \frac{\min\sigma(A)+\min\sigma(D)}{2} + \sqrt{\left(\frac{\min\sigma(A)-\min\sigma(D)}{2}\right)^2 + \nu_{\kappa+n}(BB^*)},$$

$$\lambda_n \le \frac{\max\sigma(A)+\max\sigma(D)}{2} + \sqrt{\left(\frac{\max\sigma(A)-\max\sigma(D)}{2}\right)^2 + \nu_{\kappa+n}(BB^*)}.$$

*In the particular case* $\max\sigma(D) = d < a = \min\sigma(A)$, *we have* $\kappa = 0$.

***Proof.*** For $B = 0$, the assertions are clear. For $B \ne 0$, the estimates are proved in a similar way as in the proof of Theorem 2.11.2:

Since $\|B^* \cdot\|^2$ is the closure of the quadratic form $(BB^* \cdot, \cdot)$, the classical variational principle for $BB^*$ shows that, for an arbitrary subspace $\mathcal{L} \subset \mathcal{D}(B^*)$ with $\dim\mathcal{L} = \kappa + n$, there exists an $x_{\mathcal{L}} \in \mathcal{L}$, $\|x_{\mathcal{L}}\| = 1$, with $\|B^* x_{\mathcal{L}}\|^2 \ge \nu_{\kappa+n}(BB^*)$. Hence, by Theorem 2.10.4,

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(B^*)\\ \dim\mathcal{L}=\kappa+n}} \max_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \max_{\substack{y\in\mathcal{H}_2\\ y\ne 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \ge \min_{\substack{\mathcal{L}\subset\mathcal{D}(B^*)\\ \dim\mathcal{L}=\kappa+n}} \lambda_+\begin{pmatrix} x_{\mathcal{L}}\\ B^*x_{\mathcal{L}}\end{pmatrix}.$$

Using the monotonicity of the function $f$ in (2.11.4) and then the inequality for $\|B^*x_{\mathcal{L}}\|^2$, we find that

$$\lambda_+\begin{pmatrix} x_{\mathcal{L}}\\ B^*x_{\mathcal{L}}\end{pmatrix} \ge \frac{\min\sigma(A)+\min\sigma(D)}{2} + \sqrt{\left(\frac{\min\sigma(A)-\min\sigma(D)}{2}\right)^2 + \nu_{\kappa+n}(BB^*)}\,.$$

To prove the second inequality, let $\varepsilon > 0$ be arbitrary. Then there exists a subspace $\mathcal{L}_\varepsilon \subset \mathcal{L}_{(-\infty,\nu_{\kappa+n}(BB^*)+\varepsilon]}(BB^*)$ with $\dim\mathcal{L}_\varepsilon = \kappa + n$. Again by Theorem 2.10.4, we have

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(B^*)\\ \dim\mathcal{L}=\kappa+n}} \max_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \max_{\substack{y\in\mathcal{H}_2\\ \|y\|=1}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \le \max_{\substack{x\in\mathcal{L}_\varepsilon\\ \|x\|=1}} \max_{\substack{y\in\mathcal{H}_2\\ \|y\|=1}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}.$$

Using the monotonicity of the function $f$ in (2.11.4) and then the inequality $\|B^*x\|^2 \le \nu_{\kappa+n}(BB^*) + \varepsilon$ for $x \in \mathcal{L}_\varepsilon$, we conclude that, for all $x \in \mathcal{L}_\varepsilon$, $y \in \mathcal{H}_2$, $\|x\| = \|y\| = 1$,

$$\lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \le \frac{\max\sigma(A)+\max\sigma(D)}{2} + \sqrt{\left(\frac{\max\sigma(A)-\max\sigma(D)}{2}\right)^2 + \nu_{\kappa+n}(BB^*) + \varepsilon}\,.$$

Since $\varepsilon > 0$ was arbitrary, the desired inequality follows. □

The following particular case of Theorem 2.11.5 is of interest with respect to applications, *e.g.* to matrix differential operators of Dirac type.

Here we assume that $\max\sigma(D) < 0 < \min\sigma(A)$ and we formulate the estimates not only for the positive eigenvalues of $\mathcal{A}$ (to the right of $\max\sigma(D)$), but also for the negative eigenvalues (to the left of $\min\sigma(A)$).

**Corollary 2.11.6** *Suppose that $\mathcal{A}$ is self-adjoint, $A$, $-D$ are bounded and uniformly positive, and both $BB^*$ and $B^*B$ have compact resolvents. Set*

$$a_- := \min\sigma(A), \quad a_+ := \max\sigma(A),$$
$$d_- := \min\sigma(D), \quad d_+ := \max\sigma(D),$$

*and let $\nu_n^+(BB^*)$, $\nu_n^-(B^*B)$, $n = 1, 2, \dots$, be the eigenvalues of $BB^*$, $B^*B$, respectively, enumerated non-decreasingly and counted with multiplicities. Then $(d_+, a_-) \cap \sigma(\mathcal{A}) = \emptyset$ and $\sigma(\mathcal{A})$ consists of two sequences $(\lambda_{\pm n})_{n\in\mathbb{N}}$ of eigenvalues of finite algebraic multiplicities accumulating at most at $\pm\infty$. If we enumerate $\ldots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots$ counting multiplicities, then*

$$\lambda_n \ge \frac{a_- + d_-}{2} + \sqrt{\left(\frac{a_- - d_-}{2}\right)^2 + \nu_n^+(BB^*)}, \tag{2.11.7}$$

$$\lambda_n \le \frac{a_+ + d_+}{2} + \sqrt{\left(\frac{a_+ - d_+}{2}\right)^2 + \nu_n^+(BB^*)}, \tag{2.11.8}$$

*for $n = 1, 2, \dots$ and*

$$\lambda_{-n} \ge -\frac{a_+ + d_+}{2} - \sqrt{\left(\frac{a_+ - d_+}{2}\right)^2 + \nu_n^-(B^*B)}, \tag{2.11.9}$$

$$\lambda_{-n} \le -\frac{a_- + d_-}{2} - \sqrt{\left(\frac{a_- - d_-}{2}\right)^2 + \nu_n^-(B^*B)}. \tag{2.11.10}$$

*In the limit $n \to \infty$, the asymptotic estimates*

$$\lambda_n \ge \nu_n^+(BB^*)^{1/2} + \frac{a_- + d_-}{2} + \frac{1}{2\nu_n^+(BB^*)^{1/2}}\left(\frac{a_- - d_-}{2}\right)^2 + \mathrm{O}\left(\frac{1}{\nu_n^+(BB^*)}\right),$$

$$\lambda_n \le \nu_n^+(BB^*)^{1/2} + \frac{a_+ + d_+}{2} + \frac{1}{2\nu_n^+(BB^*)^{1/2}}\left(\frac{a_+ - d_+}{2}\right)^2 + \mathrm{O}\left(\frac{1}{\nu_n^+(BB^*)}\right)$$

*hold, and analogously for the negative eigenvalues $\lambda_{-n}$.*

***Proof.*** Since $\max\sigma(D) < 0 < \min\sigma(A)$, we have $\kappa = 0$ by the last claim in Theorem 2.10.4. Now (2.11.7), (2.11.8) follow readily from Theorem 2.11.5; the latter implies, in particular, that $\lambda_+(x, y) \ge a_-$ and that $\sigma_{\mathrm{ess}}(\mathcal{A}) = \emptyset$. The estimates for the negative eigenvalues follow from Theorem 2.11.5 by considering $-\mathcal{A}$ and swapping $\mathcal{H}_1$ and $\mathcal{H}_2$. □

**Remark 2.11.7** The assumption that both $BB^*$ and $B^*B$ have compact resolvents cannot be simplified in general; only if $B$ is bijective, it is equivalent to the fact that $B$ has compact resolvent. In this case, $\sigma(BB^*) = \sigma(B^*B)$ (see [HM01, Theorem 4.2]) and hence $\nu_n^+(BB^*) = \nu_n^-(B^*B)$.

**Remark 2.11.8** If $A = a \cdot I_{\mathcal{H}_1}$, $D = d \cdot I_{\mathcal{H}_2}$ with constants $a, d \in \mathbb{R}$ in Corollary 2.11.6, then the upper and lower eigenvalue bounds in (2.11.7), (2.11.8) coincide since $a_- = a_+$, $d_- = d_+$ and so, *e.g.* if $B$ is bijective (see above),

$$\lambda_{\pm n} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + \nu_n^+(BB^*)}.$$

**Remark 2.11.9** If, in the situation of Corollary 2.11.6, there exists an element $x_0 \in \ker B^*$, $x_0 \neq 0$, we can improve the upper bound for the first positive eigenvalue $\lambda_1$ of $\mathcal{A}$; in this case, (2.10.7) yields that

$$\lambda_1 = \min_{\substack{x \in \mathcal{D}(B^*) \\ x \neq 0}} \max_{\substack{y \in \mathcal{H}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \leq \max_{\substack{y \in \mathcal{H}_2 \\ y \neq 0}} \lambda_+ \begin{pmatrix} x_0 \\ y \end{pmatrix} = \frac{(Ax_0, x_0)}{\|x_0\|^2},$$

and hence

$$a_- \leq \lambda_1 \leq \frac{(Ax_0, x_0)}{\|x_0\|^2}.$$

A typical situation where Corollary 2.11.6 applies is if $A$ and $D$ are bounded multiplication operators and $B$ is a regular differential operator.

**Example 2.11.10** Let $g \in L_\infty[0,1]$ be a positive function and consider

$$\mathcal{A} = \begin{pmatrix} g & -\mathrm{D} \\ \mathrm{D} & -g \end{pmatrix}, \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x},$$

in $L^2(0,1) \oplus L^2(0,1)$ with periodic boundary conditions in both components. Here the eigenvalues $\nu_n^+(BB^*)$, $n = 0, 1, 2, \ldots$, counted with multiplicities, are given by $\nu_0^+(BB^*) = 0$ and $\nu_n^+(BB^*) = (2k\pi)^2$ for $n = 2k-1, 2k$ and $k \in \mathbb{N}$. Hence, for $n = 2k-1, 2k$ and $k \in \mathbb{N}$, we obtain the estimates

$$\lambda_n \geq 2\pi k - \frac{\operatorname{ess\,sup} g - \operatorname{ess\,inf} g}{2} + \frac{1}{4\pi k}\left(\frac{\operatorname{ess\,sup} g + \operatorname{ess\,inf} g}{2}\right)^2 + \mathrm{O}\left(\frac{1}{k^2}\right),$$

$$\lambda_n \leq 2\pi k + \frac{\operatorname{ess\,sup} g - \operatorname{ess\,inf} g}{2} + \frac{1}{4\pi k}\left(\frac{\operatorname{ess\,sup} g + \operatorname{ess\,inf} g}{2}\right)^2 + \mathrm{O}\left(\frac{1}{k^2}\right)$$

for the positive eigenvalues of $\mathcal{A}$, and similarly for the negative eigenvalues.

**Remark 2.11.11** Self-adjoint off-diagonally dominant block operator matrices of the form

$$\mathcal{A} = \begin{pmatrix} p & -\mathrm{D}^2 + q \\ -\mathrm{D}^2 + q & r \end{pmatrix}, \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{2.11.11}$$

in $L_2[0,\infty) \oplus L_2[0,\infty)$ with real-valued functions $p$, $r$, $q$ were studied in great detail by H. Behncke and A. Fischer in [BF02], using analytical techniques like asymptotics of eigenfunctions and the $M$-matrix. Their main results concern the deficiency index and the fine structure of the spectrum including discrete parts, singular continuous and absolutely continuous part and multiplicities, as well as embedded eigenvalues.

The two-sided estimates in Corollary 2.11.6 apply to the Dirac operators on manifolds with $S^1$-symmetry considered in Example 2.5.20. In addition to the lower bound for the smallest eigenvalue in modulus derived there, we now obtain upper and lower bounds for all eigenvalues (see [KLT04, Theorem 4.17]).

**Example 2.11.12** Let $\mathcal{M} = \mathcal{B}^m \times_f \mathcal{F}^k$ be a warped product of closed spin manifolds (see Example 2.5.20) and denote by $f_{\max}$ and $f_{\min}$ the maximum and minimum, respectively, of the warp function $f\colon \mathcal{B}^m \to \mathbb{R}^+$. With the notation of Example 2.5.20, we define the dimension of the kernel of the Dirac operator $D_{\mathcal{B}}$ on the basis $\mathcal{B}$ and the Fredholm index of $D_{\mathcal{B}}^+$ and set

$$\gamma_{\mathcal{B}} := \dim\ker D_{\mathcal{B}}, \quad \delta_{\mathcal{B}} := \operatorname{ind} D_{\mathcal{B}}^+ = \dim\ker D_{\mathcal{B}}^+ - \dim\ker D_{\mathcal{B}}^-,$$
$$\alpha_{\mathcal{B}} := \frac{\gamma_{\mathcal{B}} - \delta_{\mathcal{B}}}{2} = \dim\ker D_{\mathcal{B}}^-, \quad \beta_{\mathcal{B}} := \frac{\gamma_{\mathcal{B}} + \delta_{\mathcal{B}}}{2} = \dim\ker D_{\mathcal{B}}^+.$$

Let $\zeta_n$, $n = 1, 2, \dots$, be the eigenvalues of $D_{\mathcal{B}}$ counted with multiplicities and enumerated so that the sequence $\big(|\zeta_n|\big)_1^\infty$ is non-decreasing and $\zeta_{\gamma+j} < 0$ for $j$ odd. With $B_\Lambda$ as in (2.5.9), the eigenvalues $\nu_{\Lambda,n}^+$ and $\nu_{\Lambda,n}^-$, $n = 1, 2, \dots$, of the operator $(B_\Lambda B_\Lambda^*)^{1/2}$ and of $(B_\Lambda^* B_\Lambda)^{1/2}$, respectively, enumerated non-decreasingly, for even $m$ are given by

$$\begin{aligned}
\nu_{\Lambda,n}^+ &= 0, & n &= 1, \dots, \alpha_{\mathcal{B}} r(\Lambda),\\
\nu_{\Lambda,n}^+ &= \zeta_{\gamma+2j}, & n &= \alpha_{\mathcal{B}} r(\Lambda) + (j-1)r(\Lambda) + 1, \dots, \alpha_{\mathcal{B}} r(\Lambda) + jr(\Lambda),\\
\nu_{\Lambda,n}^- &= 0, & n &= 1, \dots, \beta_{\mathcal{B}} r(\Lambda),\\
\nu_{\Lambda,n}^- &= \nu_{1,\Lambda,n+\delta r(\Lambda)}, & n &= \beta_{\mathcal{B}} r(\Lambda) + 1, \beta_{\mathcal{B}} r(\Lambda) + 2, \dots,
\end{aligned}$$

and for odd $m$ by

$$\nu_{\Lambda,n}^+ = \nu_{\Lambda,n}^- = |\zeta_j|, \quad n = (j-1)r(\Lambda) + 1, \dots, jr(\Lambda).$$

The eigenvalues $\lambda_{\Lambda,n}$, $n = \pm1, \pm2, \dots$, of $D_{\mathcal{M}}$ of weight $\Lambda \neq 0$, $\Lambda \in \sigma(D_{\mathcal{F}})$, which coincide with the eigenvalues of the block operator matrix $\mathcal{A}_\Lambda$ given by (2.5.10), can be enumerated so that $\cdots \le \lambda_{\Lambda,-2} \le \lambda_{\Lambda,-1} < 0 < \lambda_{\Lambda,1} \le \lambda_{\Lambda,2} \le \cdots$. Now Corollary 2.11.6 yields that, for $n = 1, 2, \dots$,

$$\lambda_{\Lambda,\pm n} \geq -\frac{|\Lambda|}{2}\Big(\frac{1}{f_{\min}}-\frac{1}{f_{\max}}\Big) \pm \sqrt{\frac{\Lambda^2}{4}\Big(\frac{1}{f_{\min}}+\frac{1}{f_{\max}}\Big)^2+\big(\nu_{\Lambda,n}^{\pm}\big)^2},$$
$$\lambda_{\Lambda,\pm n} \leq \frac{|\Lambda|}{2}\Big(\frac{1}{f_{\min}}-\frac{1}{f_{\max}}\Big) \pm \sqrt{\frac{\Lambda^2}{4}\Big(\frac{1}{f_{\min}}+\frac{1}{f_{\max}}\Big)^2+\big(\nu_{\Lambda,n}^{\pm}\big)^2}.$$

If $\Lambda = 0$ in the above estimates, upper and lower bounds are the same and coincide with the eigenvalues of weight 0, apart from multiplicities.

Note that, for even $m$, the spectrum of weight $\Lambda$ is not symmetric to 0; if, however, the spectrum of the fibre $\mathcal{F}$ is symmetric to 0, then so is the spectrum of the whole Dirac operator $D_{\mathcal{M}}$. In fact, if $\lambda \in \sigma_{\rm p}(D_{\mathcal{M}})$ is an eigenvalue of $\mathcal{A}_\Lambda$ with eigenvector $(\psi_1\, \psi_2)^{\rm t}$, then $-\lambda$ is an eigenvalue of $\mathcal{A}_{-\Lambda}$ with eigenvector $(\psi_1\ -\psi_2)^{\rm t}$ since $-A_{i,\Lambda} = A_{i,-\Lambda}$, $i=1,2$, and so $-\lambda \in \sigma_{\rm p}(D_{\mathcal{M}})$.

If we further estimate the bounds for $\lambda_{\Lambda,\pm n}$ by omitting the terms $(\nu_{\Lambda,n}^{\pm})^2$, we obtain that the eigenvalue $\lambda_{\Lambda,\min}$ of weight $\Lambda \neq 0$ with smallest modulus satisfies the lower estimate $|\lambda_{\Lambda,\min}| \geq |\Lambda|/f_{\max}$ already proved in (2.5.11); if $0 \in \sigma(D_{\mathcal{B}})$, then either $\nu_{\Lambda,1}^{+} = 0$ or $\nu_{\Lambda,1}^{-} = 0$, and hence, altogether, $\lambda_{\Lambda,\min}$ satisfies the two-sided estimate

$$\frac{|\Lambda|}{f_{\max}} \leq |\lambda_{\Lambda,\min}| \leq \frac{|\Lambda|}{f_{\min}}.$$

In the special case of a Riemannian spin manifold $\mathcal{B}$ with parallel spinor $\Psi$, *i.e.* $\nabla\Psi = 0$, the estimate for the first positive eigenvalue $\lambda_{\Lambda,1}$ of weight $\Lambda$ of the Dirac operator on $\mathcal{B}\times_f\mathcal{F}$ is improved by Remark 2.11.9 to

$$\frac{|\Lambda|}{f_{\max}} \leq \lambda_{\Lambda,1} \leq \frac{1}{\operatorname{vol}\mathcal{B}}\int_{\mathcal{B}} \frac{|\Lambda|}{f}\,{\rm d}\mathcal{B}.$$

Examples of such manifolds are the warped products $\mathcal{M} = S^1\times_f\mathcal{F}$ with a closed manifold $\mathcal{F}$; here the basis manifold is one-dimensional and the estimates may be improved by ODE methods like Floquet theory (see [Kra03]).

Finally, we derive eigenvalue estimates for $\mathcal{J}$-self-adjoint block operator matrices; we restrict ourselves to Theorem 2.10.8, *i.e.* to the case where the quadratic numerical range is real and hence $\kappa = 0$.

**Theorem 2.11.13** *Suppose that $A = A^*$ is bounded from below, $D = D^*$ is bounded from above, $\max\sigma(D) =: d < a := \min\sigma(A)$, $B$ is closed, $C = -B^*$, and $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$, $\mathcal{D}(|D|^{1/2}) \subset \mathcal{D}(B)$ (so that $\mathcal{A}$ satisfies condition* (a) *in Proposition* 2.10.1 *and is $\mathcal{J}$-self-adjoint). Assume that $W^2(\mathcal{A}) \subset \mathbb{R}$, that there is a $\gamma \in \mathbb{R}$ with $\overline{\Lambda_-(\mathcal{A})} < \gamma < \overline{\Lambda_+(\mathcal{A})}$, and define $\lambda_{\rm e} := \min\big(\sigma_{\rm ess}(\mathcal{A}) \cap (\gamma,\infty)\big)$. Let $\nu_1(A) \leq \nu_2(A) \leq \ldots \leq \nu_M(A)$,*

*$M \in \mathbb{N}_0 \cup \{\infty\}$, be the eigenvalues of $A$ below $\sigma_{\rm ess}(A)$ counted with multiplicities, set $\nu_k(A) := \min \sigma_{\rm ess}(A)$ for $k > M$, and let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, be the finite or infinite sequence of eigenvalues of $\mathcal{A}$ in $(\gamma, \lambda_{\rm e})$ counted with multiplicities. If $D$ is bounded, then, for $n = 1, \ldots, N$,*

$$\lambda_n \le \frac{\nu_n(A)+\min\sigma(D)}{2} + \sqrt{\left(\frac{\nu_n(A)-\min\sigma(D)}{2}\right)^2 - \min\sigma(BB^*)}.$$

*If $B$ is bounded, $\|B\| \le \dfrac{\min\sigma(A) - \max\sigma(D)}{2}$, then, for $n = 1, \ldots, N$,*

$$\lambda_n \ge \frac{\nu_n(A) + \max\sigma(D)}{2} + \sqrt{\left(\frac{\nu_n(A) - \max\sigma(D)}{2}\right)^2 - \|B\|^2}.$$

***Proof.*** For $\beta \ge 0$, $s > t$, and $(s-t)^2 \ge \beta$, the function

$$f(s,t) := s + t + \sqrt{(s-t)^2 - \beta}$$

is increasing in $s$ and decreasing in $t$. For every $\varepsilon > 0$, there exists a subspace $\mathcal{L}_\varepsilon \subset \mathcal{L}_{(-\infty,\nu_n(A)+\varepsilon]}(A)$ with $\dim \mathcal{L}_\varepsilon = n$. Then, by Theorem 2.10.8,

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=n}} \max_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \min_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \le \max_{\substack{x\in\mathcal{L}_\varepsilon\\ \|x\|=1}} \min_{\substack{y\in\mathcal{H}_2\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \le \max_{\substack{x\in\mathcal{L}_\varepsilon\\ \|x\|=1}} \lambda_+\begin{pmatrix} x\\ B^*x\end{pmatrix}.$$

From the monotonicity of $f$ (note that the condition $(s-t)^2 \ge \beta$ is fulfilled because $W^2(\mathcal{A})$ is real) together with the inequalities $(Ax,x) \le \nu_n(A) + \varepsilon$, $(DB^*x, B^*x)/\|B^*x\|^2 \ge \min\sigma(D)$, $x \in \mathcal{L}_\varepsilon$, it follows that, for $x \in \mathcal{L}_\varepsilon$,

$$\begin{aligned}
&\lambda_+\begin{pmatrix} x\\ B^*x\end{pmatrix}\\
&= \frac{(Ax,x)}{2} + \frac{(DB^*x,B^*x)}{2\|B^*x\|^2} + \sqrt{\left(\frac{(Ax,x)}{2} - \frac{(DB^*x,B^*x)}{2\|B^*x\|^2}\right)^2 - (BB^*x,x)}\\
&\le \frac{\nu_n(A)+\varepsilon+\min\sigma(D)}{2} + \sqrt{\left(\frac{\nu_n(A)+\varepsilon-\min\sigma(D)}{2}\right)^2 - \min\sigma(BB^*)};
\end{aligned}$$

since $\varepsilon > 0$ was arbitrary, this proves the upper estimate for $\lambda_n$.

For the proof of the lower estimate, let $\mathcal{L} \subset \mathcal{D}(A)$ with $\dim\mathcal{L} = n$ be an arbitrary subspace. By the classical variational principle for the semibounded operator $A$, there exists an $x_{\mathcal{L}} \in \mathcal{L}$ with $\|x_{\mathcal{L}}\| = 1$ such that $(Ax_{\mathcal{L}}, x_{\mathcal{L}}) \ge \nu_n$. Then, again by Theorem 2.10.8,

$$\begin{aligned}
\lambda_n &= \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=n}} \max_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \min_{\substack{y\in\mathcal{D}(D)\\ \|y\|=1}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \ge \min_{\substack{\mathcal{L}\subset\mathcal{D}(A)\\ \dim\mathcal{L}=n}} \min_{\substack{y\in\mathcal{D}(D)\\ \|y\|=1}} \lambda_+\begin{pmatrix} x_{\mathcal{L}}\\ y\end{pmatrix}\\
&\ge \frac{\nu_n(A)+\max\sigma(D)}{2} + \sqrt{\left(\frac{\nu_n(A)-\max\sigma(D)}{2}\right)^2 - \|B\|^2}. \qquad \square
\end{aligned}$$

We conclude this section by applying the above results to estimate the eigenvalues of a $\lambda$-rational spectral problem for the Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^m$. In the one-dimensional case, which is the $\lambda$-rational Sturm-Liouville problem considered in Example 2.4.3, the eigenvalues were investigated in many papers, see *e.g.* [AL95], [GM03], [MSS98], [Lut99], and the references therein.

**Example 2.11.14** Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, $q, w, u \in L^\infty(\Omega)$, and assume that either $w \geq 0$ or $w \leq 0$. We consider the spectral problem

$$\Big(-\Delta + q - \lambda - \frac{w}{u-\lambda}\Big)\varphi = 0, \quad \varphi|_{\partial\Omega} = 0. \tag{2.11.12}$$

As in Example 2.4.3, we see that (2.11.12) is equivalent to the eigenvalue problem for the block operator matrix

$$\mathcal{A} = \begin{pmatrix} -\Delta + q & \sqrt{w} \\ (\operatorname{sign} w)\sqrt{w} & u \end{pmatrix}$$

in $L^2(\Omega) \oplus L^2(\Omega)$ with domain $\mathcal{D}(\mathcal{A}) = \{y_1 \in W_2^2(\Omega) : y_1|_{\partial\Omega} = 0\} \oplus L^2(\Omega)$; in fact, the equation (2.11.12) is exactly the spectral problem $S_1(\lambda)\varphi = 0$ for the first Schur complement $S_1$ of $\mathcal{A}$. We distinguish two cases:

a) $w \geq 0$: Here $\mathcal{A}$ is self-adjoint and satisfies all assumptions of Theorem 2.11.2 with $b' = b'' = 0$, $a' = \operatorname{ess\,sup} w$, $a'' = \operatorname{ess\,inf} w$ (see Remark 2.11.3). Let $\nu_1 \leq \nu_2 \leq \cdots$ be the eigenvalues of the left upper corner $A$ of $\mathcal{A}$, *i.e.* of the spectral problem

$$\big(-\Delta + q - \lambda\big)\varphi = 0, \quad \varphi|_{\partial\Omega} = 0. \tag{2.11.13}$$

If $\lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of (2.11.12) greater than $\operatorname{ess\,sup} u$ and $\kappa$ is the number of negative eigenvalues of $S_1(\gamma) = -\Delta + q - \gamma - w/(u-\gamma)$ for a fixed $\gamma \in \big(\operatorname{ess\,sup} u, \lambda_1\big)$, then, for $n = 1, 2, \dots,$

$$\begin{aligned} \lambda_n &\leq \frac{\nu_{\kappa+n} + \operatorname{ess\,sup} u}{2} + \sqrt{\Big(\frac{\nu_{\kappa+n} - \operatorname{ess\,sup} u}{2}\Big)^2 + \operatorname{ess\,sup} w}, \\ \lambda_n &\geq \frac{\nu_{\kappa+n} + \operatorname{ess\,inf} u}{2} + \sqrt{\Big(\frac{\nu_{\kappa+n} - \operatorname{ess\,inf} u}{2}\Big)^2 + \operatorname{ess\,inf} w}. \end{aligned} \tag{2.11.14}$$

For the one-dimensional case, the first inequality in (2.11.14) was proved in [Lan00, Corollary 4.2] by different means for $q = 0$. In [EL04], using the variational principle for the Schur complement, it was shown that

$$\lambda_n \leq \nu_{\kappa+n} + \frac{\operatorname{ess\,sup} w}{\nu_{\kappa+n} - \operatorname{ess\,sup} u};$$

however, due to the inequality

$$\frac{s+t}{2} + \sqrt{\Big(\frac{s-t}{2}\Big)^2 + \beta} \le s + \frac{\beta}{s-t}, \quad s > t,\ \beta \ge 0,$$

this estimate is weaker than the estimate from above in (2.11.14).

b) $w \le 0$: Here $\mathcal{A}$ is $\mathcal{J}$-self-adjoint. If we suppose that

$$\operatorname{ess\,sup} \sqrt{|w|} < \frac{1}{2}\big(\min \sigma(-\Delta + q) - \operatorname{ess\,sup} u\big),$$

then $\mathcal{A}$ satisfies the assumptions of Theorem 2.11.13; in particular, its quadratic numerical range and spectrum are real. Let again $\nu_1 \le \nu_2 \le \dots$ be the eigenvalues of (2.11.13). If $\lambda_1 \le \lambda_2 \le \dots$ are the eigenvalues of (2.11.12) greater than $\big(\min \sigma(-\Delta+q)+\operatorname{ess\,sup} u\big)/2$, then, for $n = 1, 2, \dots,$

$$\lambda_n \le \frac{\nu_n + \operatorname{ess\,inf} u}{2} + \sqrt{\Big(\frac{\nu_n - \operatorname{ess\,inf} u}{2}\Big)^2 - \operatorname{ess\,inf} |w|},$$

$$\lambda_n \ge \frac{\nu_n + \operatorname{ess\,sup} u}{2} + \sqrt{\Big(\frac{\nu_n - \operatorname{ess\,sup} u}{2}\Big)^2 - \operatorname{ess\,sup} |w|}.$$

Applications of the eigenvalue estimates derived in this section are given in Section 3.1 to a spectral problem from magnetohydrodynamics and in Section 3.2 to the eigenvalue problem for the angular part of a Dirac operator in curved space-time.

# Chapter 3

# Applications in Mathematical Physics

Systems of linear partial differential equations of mixed order and type arise frequently in mathematical physics, *e.g.* in linear stability problems in magnetohydrodynamics and fluid mechanics, as well as in quantum mechanics. Such systems may be described by block operator matrices whose entries are differential operators. This opens up the way to use the spectral properties of parts of the system (the entries of the block operator matrix) to obtain spectral information about the whole system (the block operator matrix).

In this chapter we apply the results of Chapter 2 to three different examples: to an upper dominant block operator matrix arising as a force operator in magnetohydrodynamics, to a diagonally dominant block operator matrix governing the linear stability of the Ekman boundary layer in fluid mechanics, and to the prototype of an off-diagonally dominant block operator matrix, the Dirac operator from quantum mechanics.

## 3.1 Upper dominant block operator matrices in magnetohydrodynamics

The linearized equations of ideal magnetohydrodynamics (MHD) are used to describe some phenomena occurring in the interaction of a magnetoactive plasma and an exterior magnetic field (see [Lif89]). These equations are systems of partial differential equations of mixed order and type (Douglis-Nirenberg elliptic systems); the corresponding linear operator, also called force operator, is an upper dominant block operator matrix which is typically essentially self-adjoint. If the spectrum of the force operator is nonnegative, then the plasma configuration is stable; knowledge about the essential spectrum of the force operator may be used to heat the plasma by means of so-called Alfvén waves (see [Raĭ91], [Lif85]). The first rigorous mathematical approach to calculate the essential spectrum of block operator matrices occurring in MHD was based on pseudo-differential calculus and is due to G. Grubb and G. Geymonat (see *e.g.* [GG77], [GG79]) and J. Descloux and G. Geymonat (see [DG79], [DG80]). They seem to be the first who observed that, unlike regular differential operators, regular *matrix* differential operators may have non-empty essential spectrum.

In this section, we present an operator theoretic approach to investigate the spectrum of the force operator associated with a plane equilibrium configuration of a gravitating ideal plasma in an ambient magnetic field (see [LL84, §68], [Goe83], [Lif89]). We apply the results of Sections 2.4 and 2.11 to determine the essential spectrum and to estimate the eigenvalues of this problem (see [Raĭ91], [ALMS94], [AMS98], [KLT04, Example 4.5]).

**Example 3.1.1** The linear stability theory of small oscillations of a magnetized gravitating plane equilibrium layer of a hot compressible ideal plasma, bounded by rigid perfectly conducting planes (at $x=0$ and $x=1$, say), leads to a spectral problem for a system of three coupled differential equations on the interval $[0,1]$. The corresponding linear operator is a $(1{+}2)\times(1{+}2)$ block operator matrix given by (see *e.g.* [ALMS94, Section 5], [Lif89, Chapter 7.3, (3.19)], and [Raĭ91])

$$\mathcal{A}=\left(\begin{array}{c|cc}\rho_0^{-1}\mathrm{D}\rho_0(v_\mathrm{a}^2+v_\mathrm{s}^2)\mathrm{D}+k^2v_\mathrm{a}^2 & \big(\rho_0^{-1}\mathrm{D}\rho_0(v_\mathrm{a}^2+v_\mathrm{s}^2)+\mathrm{i}g\big)k_\perp & (\rho_0^{-1}\mathrm{D}\rho_0v_\mathrm{s}^2+\mathrm{i}g)k_\parallel \\ \hline k_\perp\big((v_\mathrm{a}^2+v_\mathrm{s}^2)\mathrm{D}-\mathrm{i}g\big) & k^2v_\mathrm{a}^2+k_\perp^2v_\mathrm{s}^2 & k_\perp k_\parallel v_\mathrm{s}^2 \\ k_\parallel(v_\mathrm{s}^2\mathrm{D}-\mathrm{i}g) & k_\perp k_\parallel v_\mathrm{s}^2 & k_\parallel^2 v_\mathrm{s}^2\end{array}\right)$$

on $[0,1]$ with Dirichlet boundary conditions for the first component at $x=0$ and $x=1$. Here $\mathrm{D}=-\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ is a first order derivative, $g$ is the gravitational constant, and all other coefficients are functions on $[0,1]$: $\rho_0$ the equilibrium density of the plasma, $v_\mathrm{a}$ the Alfvén speed, $v_\mathrm{s}$ the sound speed, $k_\perp$, $k_\parallel$ the coordinates of the wave vector $\mathbf{k}$ with respect to the field allied orthonormal bases, and $k=\sqrt{k_\perp^2+k_\parallel^2}$ the length of $\mathbf{k}$.

The above block operator matrix $\mathcal{A}$ is considered in the Hilbert space product $L^2_{\rho_0}(0,1)\oplus\big(L^2_{\rho_0}(0,1)\big)^2$ where $L^2_{\rho_0}(0,1)$ denotes the $L^2$-space on $[0,1]$ with weight $\rho_0$. If $W^{k,2}_{\rho_0}(0,1)$ and $W^{k,2}_{\rho_0,0}(0,1)$ denote the Sobolev space of order $k$ associated with $L^2_{\rho_0}(0,1)$ without and with Dirichlet boundary conditions, respectively, then the entries $A$, $B$, $C$, and $D$ of $\mathcal{A}$ are given by

$$A:=\rho_0^{-1}\mathrm{D}\rho_0(v_\mathrm{a}^2+v_\mathrm{s}^2)\mathrm{D}+k^2v_\mathrm{a}^2, \qquad \mathcal{D}(A):=\big(W^{2,2}_{\rho_0}(0,1)\cap W^{1,2}_{\rho_0,0}(0,1)\big),$$

$$B:=\Big(\big(\rho_0^{-1}\mathrm{D}\rho_0(v_\mathrm{a}^2+v_\mathrm{s}^2)+\mathrm{i}g\big)k_\perp\ \ (\rho_0^{-1}\mathrm{D}\rho_0v_\mathrm{s}^2+\mathrm{i}g)k_\parallel\Big),\ \ \mathcal{D}(B):=\big(W^{1,2}_{\rho_0}(0,1)\big)^2,$$

$$C := \begin{pmatrix} k_\perp\big((v_{\rm a}^2+v_{\rm s}^2){\rm D}-{\rm i}g\big) \\ k_\|(v_{\rm s}^2{\rm D}-{\rm i}g) \end{pmatrix}, \qquad \mathcal{D}(C) := W^{1,2}_{\rho_0,0}(0,1)\,,$$

$$D := \begin{pmatrix} k^2v_{\rm a}^2+k_\perp^2v_{\rm s}^2 & k_\perp k_\| v_{\rm s}^2 \\ k_\perp k_\| v_{\rm s}^2 & k_\|^2 v_{\rm s}^2 \end{pmatrix}, \quad \mathcal{D}(D) := \big(L^2_{\rho_0}(0,1)\big)^2.$$

**Proposition 3.1.2** *The block operator matrix $\mathcal{A}$ with its domain*

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A)\oplus\mathcal{D}(B) = \big(W^{2,2}_{\rho_0}(0,1)\cap W^{1,2}_{\rho_0,0}(0,1)\big)\oplus\big(W^{1,2}_{\rho_0}(0,1)\big)^2$$

*is upper dominant and essentially self-adjoint.*

***Proof.*** It is not difficult to see that $A$ is an unbounded positive self-adjoint operator in $L^2_{\rho_0}(0,1)$, $D$ is a bounded self-adjoint operator in $\big(L^2_{\rho_0}(0,1)\big)^2$, $B$ is closed, and $C=B^*$. Since $\mathcal{D}(A)\subset\mathcal{D}(B^*)$, $\mathcal{D}(B)\subset\mathcal{D}(D)$, the block operator matrix $\mathcal{A}$ is upper dominant by Remark 2.2.2 iii). Furthermore, $\mathcal{D}(B)\cap\mathcal{D}(D) = \big(W^{1,2}_{\rho_0}(0,1)\big)^2$ is dense in $\big(L^2_{\rho_0}(0,1)\big)^2$ and hence a core of $D$, and it can be shown that $\mathcal{D}(A^{1/2})\subset\mathcal{D}(B^*)$ (see *e.g.* [ALMS94, Proposition 4.1]). Thus $\mathcal{A}$ is essentially self-adjoint by Proposition 2.3.6. □

The self-adjoint closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ is called *force operator*. Next we determine the essential spectrum of $\overline{\mathcal{A}}$ using the second Schur complement.

**Theorem 3.1.3** *The essential spectrum of the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ is the set*

$$\sigma_{\rm ess}\big(\overline{\mathcal{A}}\big) = \lambda_{{\rm a},k}\big([0,1]\big)\cup\lambda_{{\rm m},k}\big([0,1]\big),$$

*the functions $\lambda_{{\rm a},k}, \lambda_{{\rm m},k}$ being the squares of the Alfvén and mean frequencies:*

$$\lambda_{{\rm a},k} := k_\|^2 v_{\rm a}^2, \quad \lambda_{{\rm m},k} := k_\|^2\frac{v_{\rm a}^2v_{\rm s}^2}{v_{\rm a}^2+v_{\rm s}^2}.$$

***Proof.*** We calculate the essential spectrum of $\overline{\mathcal{A}}$ in two steps. First we use Corollary 2.4.13. Obviously, $A$ is positive, in particular, $0\in\rho(A)$, and $A$ has compact resolvent. This implies that $A^{-1/2}$ is compact and hence so is $D^{-1}B^*A^{-1} = D^{-1}B^*A^{-1/2}A^{-1/2}$ because $D^{-1}$ and $B^*A^{-1/2}$ are bounded. Hence $\mathcal{A}$ satisfies all assumptions of Corollary 2.4.13, which yields

$$\sigma_{\rm ess}\big(\overline{\mathcal{A}}\big) = \sigma_{\rm ess}\big(\overline{S_2(0)}\big) = \sigma_{\rm ess}\big(D-\overline{B^*A^{-1}B}\big).$$

Now we employ Theorem 2.4.15 to calculate $\sigma_{\rm ess}\big(D-\overline{B^*A^{-1}B}\big)$ by splitting off the lower order terms in $A$, $B$, $C=B^*$, and $D$. To this end, set

$$A_0 := \rho_0^{-1}{\rm D}\rho_0(v_{\rm a}^2+v_{\rm s}^2){\rm D}, \qquad \mathcal{D}(A_0) := \mathcal{D}(A),$$

$$B_0 := \Big(\rho_0^{-1}{\rm D}\rho_0(v_{\rm a}^2+v_{\rm s}^2)k_\perp \;\; \rho_0^{-1}{\rm D}\rho_0 v_{\rm s}^2 k_\|\Big), \quad \mathcal{D}(B_0) := \mathcal{D}(B),$$

$$C_0 := \begin{pmatrix} k_\perp (v_{\rm a}^2 + v_{\rm s}^2)\mathrm{D} \\ k_\| v_{\rm s}^2 \mathrm{D} \end{pmatrix} = B_0^*, \mathcal{D}(C_0) := \mathcal{D}(C) = \mathcal{D}(B_0^*).$$

Then all assumptions of Theorem 2.4.15 are satisfied with $\mu_0 = 0$ and thus

$$\sigma_{\rm ess}\big(\overline{\mathcal{A}}\big) = \sigma_{\rm ess}\big(D - \overline{B_0^* A_0^{-1} B_0}\big).$$

The bounded operator $B_0^* A_0^{-1} B_0$ on $\mathcal{D}(B_0) = \big(W_{\rho_0}^{1,2}(0,1)\big)^2$ has the form

$$\begin{aligned} &\begin{pmatrix} k_\perp (v_{\rm a}^2 + v_{\rm s}^2) \\ k_\| v_{\rm s}^2 \end{pmatrix} \mathrm{D}\Big(\rho_0^{-1}\mathrm{D}\rho_0 (v_{\rm a}^2 + v_{\rm s}^2)\mathrm{D}\Big)^{-1} \big(\rho_0^{-1}\mathrm{D}\rho_0\big) \Big( (v_{\rm a}^2 + v_{\rm s}^2) k_\perp \;\; v_{\rm s}^2 k_\| \Big) \\ &= \begin{pmatrix} k_\perp (v_{\rm a}^2 + v_{\rm s}^2) \\ k_\| v_{\rm s}^2 \end{pmatrix} \frac{1}{v_{\rm a}^2 + v_{\rm s}^2} \Big( (v_{\rm a}^2 + v_{\rm s}^2) k_\perp \;\; v_{\rm s}^2 k_\| \Big) + K_0 \\ &= \begin{pmatrix} k_\perp^2 (v_{\rm a}^2 + v_{\rm s}^2) & k_\perp k_\| v_{\rm s}^2 \\ k_\perp k_\| v_{\rm s}^2 & k_\|^2 v_{\rm s}^4 / (v_{\rm a}^2 + v_{\rm s}^2) \end{pmatrix} + K_0 \end{aligned}$$

with a compact integral operator $K_0$. Together with the formula for the matrix multiplication operator $D$ and the relation $k^2 = k_\perp^2 + k_\|^2$, we obtain

$$\begin{aligned} \sigma_{\rm ess}\big(\overline{\mathcal{A}}\big) &= \sigma_{\rm ess}\left(\begin{pmatrix} k_\|^2 v_{\rm a}^2 & 0 \\ 0 & k_\|^2 v_{\rm a}^2 v_{\rm s}^2 / (v_{\rm a}^2 + v_{\rm s}^2) \end{pmatrix}\right) = \sigma_{\rm ess}\left(\begin{pmatrix} \lambda_{{\rm a},k} & 0 \\ 0 & \lambda_{{\rm m},k} \end{pmatrix}\right) \\ &= \lambda_{{\rm a},k}\big([0,1]\big) \cup \lambda_{{\rm m},k}\big([0,1]\big). \end{aligned}$$

□

Next we study the discrete spectrum of the force operator $\overline{\mathcal{A}}$ using the eigenvalue estimates proved in Section 2.11.

**Theorem 3.1.4** *Define the functions $f_\pm : [0,1] \to \mathbb{R}$ by*

$$f_\pm := \frac{k^2 (v_{\rm a}^2 + v_{\rm s}^2)}{2} \pm \sqrt{\frac{k^4 (v_{\rm a}^2 + v_{\rm s}^2)^2}{4} - k^2 k_\|^2 v_{\rm a}^2 v_{\rm s}^2}.$$

*Then the spectrum of the force operator $\overline{\mathcal{A}}$ in the interval $\big(\max f_+, \infty\big)$ consists of eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ accumulating at $\infty$. If $\nu_1 \le \nu_2 \le \cdots$ are the eigenvalues of the operator*

$$A = \rho_0^{-1}\mathrm{D}\rho_0 (v_{\rm a}^2 + v_{\rm s}^2)\mathrm{D} + k^2 v_{\rm a}^2, \quad \mathcal{D}(A) = \big(W_{\rho_0}^{2,2}(0,1) \cap W_{\rho_0,0}^{1,2}(0,1)\big),$$

*in $L_{\rho_0}^2(0,1)$ and the constant $\kappa$ is given by* (2.10.14) (*see also* (2.10.14)), *then the eigenvalues of $\overline{\mathcal{A}}$ satisfy*

$$\lambda_n \le \nu_{\kappa+n} + \max \frac{(v_{\rm a}^2 + v_{\rm s}^2)^2 k_\perp^2 + v_{\rm s}^4 k_\|^2}{v_{\rm a}^2 + v_{\rm s}^2} + \mathrm{O}\Big(\frac{1}{\nu_{\kappa+n}}\Big),$$

$$\lambda_n \ge \nu_{\kappa+n} + \min \frac{(v_{\rm a}^2 + v_{\rm s}^2)^2 k_\perp^2 + v_{\rm s}^4 k_\|^2}{v_{\rm a}^2 + v_{\rm s}^2} + \mathrm{O}\Big(\frac{1}{\nu_{\kappa+n}}\Big).$$

***Proof.*** First we show that $\mathcal{A}$ satisfies the assumptions of Theorem 2.11.2 and determine $a'$, $a''$, $b'$, $b''$ so that (2.11.2), (2.11.3) hold. Using the definition of the operators $A$ and $B^* = C$ and integrating by parts, we find

$$\mathfrak{a}[y] = \int_0^1 \rho_0\, p_1 |y'|^2 \,\mathrm{d}x + \int_0^1 \rho_0\, q_1 |y|^2 \,\mathrm{d}x,$$
$$\|B^* y\|^2 = \int_0^1 \rho_0\, p_2 |y'|^2 \,\mathrm{d}x + \int_0^1 \rho_0\, q_2 |y|^2 \,\mathrm{d}x;$$

here the functions $p_1$, $q_1$, $p_2$, and $q_2$ are given by

$$p_1 := v_\mathrm{a}^2 + v_\mathrm{s}^2, \qquad q_1 := k^2 v_\mathrm{a}^2,$$
$$p_2 := (v_\mathrm{a}^2 + v_\mathrm{s}^2)^2 k_\perp^2 + v_\mathrm{s}^4 k_\parallel^2, \quad q_2 := k^2 g^2 - \frac{g}{\rho_0}\Bigl(\rho_0 \Bigl((v_\mathrm{a}^2 + v_\mathrm{s}^2) k_\perp^2 + v_\mathrm{s}^2 k_\parallel^2\Bigr)\Bigr)'.$$

Then the assumptions (2.11.2), (2.11.3) of Theorem 2.11.2 hold with

$$b' = \max \frac{p_2}{p_1}, \quad a' = \max\bigl\{\max q_2 - b \min q_1, 0\bigr\},$$
$$b'' = \min \frac{p_2}{p_1}, \quad a'' = \min q_2 - b' \max q_1;$$

for instance, (2.11.2) follows from

$$\begin{aligned}\|B^* y\|^2 &\le b' \Bigl(\int_0^1 \rho_0\, p_1 |y'|^2 \,\mathrm{d}x + \int_0^1 \rho_0\, q_1 |y|^2 \,\mathrm{d}x\Bigr) + \int_0^1 \rho_0 \bigl(-b q_1 + q_2\bigr) |y|^2 \,\mathrm{d}x \\ &\le b'\, \mathfrak{a}[y] + a' \|y\|^2.\end{aligned}$$

Finally, the spectrum of the $2\times 2$ matrix multiplication operator $D$ in right lower corner of the block operator matrix $\mathcal{A}$ is the range of the functions $f_\pm$ defined above (note that, for fixed $x$, $f_\pm(x)$ are the eigenvalues of the $2 \times 2$ matrix $D(x)$), and thus

$$\max \sigma(D) = \max f_+, \quad \min \sigma(D) = \min f_-.$$

The claimed estimates now follow from Corollary 2.11.4 of Theorem 2.11.2 if we only use the first two terms in the asymptotic estimates (2.11.5) and (2.11.6) therein. □

This application illustrates how the two Schur complements may be used to investigate different spectral properties of a given block operator matrix: Whereas the essential spectrum of the force operator $\overline{\mathcal{A}}$ in Example 3.1.1 was determined by means of the second Schur complement, the eigenvalue estimates used for the discrete spectrum of $\overline{\mathcal{A}}$ were based on variational principles for the first Schur complement.

## 3.2 Diagonally dominant block operator matrices in fluid mechanics

Linear stability in hydrodynamics is often determined by non-self-adjoint spectral problems which, in addition, may have the non-standard form $\mathcal{A}x = \lambda\mathcal{B}x$ with linear operators $\mathcal{A}$ and $\mathcal{B}$. A typical example is the famous Orr-Sommerfeld equation describing the linear stability of a flow of a viscous incompressible fluid; in this case, the operators $\mathcal{A}$ and $\mathcal{B}$ are ordinary differential operators or order four and two, respectively, equipped with suitable boundary conditions. On a compact interval, the Orr-Sommerfeld problem has discrete spectrum, on the singular interval $[0,\infty)$, it also has essential spectrum on a half-line parallel to the real axis.

Another example is the linear stability for the two-dimensional Ekman boundary layer flow which is produced in a rotating tank with small inflow (see [Lil66]). Here the linear operators $\mathcal{A}$ and $\mathcal{B}$ are diagonally dominant block operator matrices whose entries are non-self-adjoint ordinary differential operators on $[0,\infty)$. In fact, the left upper equation of the $2\times 2$ matrix equality $\mathcal{A}x = \lambda\mathcal{B}x$ coincides with the Orr-Sommerfeld equation.

The essential spectrum for the Ekman boundary layer problem was rigorously determined in [GM04] by explicitly constructing singular sequences, though a heuristic argument of [Spo82] already indicated the result earlier. In this section we demonstrate how the method presented in Section 2.4 facilitates the calculation of the essential spectrum (see [MT94, Theorem 3.1]).

**Example 3.2.1** The stability theory of Ekman boundary layers leads to the system of differential equations (see [Fal63], [Lil66], [Gre80, Section 6.4])

$$\begin{pmatrix} (-\mathrm{D}^2+\alpha^2)^2+\mathrm{i}\alpha RV(-\mathrm{D}^2+\alpha^2)+\mathrm{i}\alpha RV'' & 2\mathrm{D} \\ 2\mathrm{D}+\mathrm{i}\alpha RU' & -\mathrm{D}^2+\alpha^2+\mathrm{i}\alpha RV \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} -\mathrm{D}^2+\alpha^2 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{3.2.1}$$

on the interval $[0,\infty)$ with boundary conditions

$$y_1(0) = y_1'(0) = y_2(0) = 0, \quad y_1(\infty) = y_1'(\infty) = y_2(\infty) = 0; \tag{3.2.2}$$

here $\mathrm{D} = \mathrm{d}/\mathrm{d}x$ is the derivative with respect to the independent variable $x \in [0,\infty)$. The constant $R \geq 0$ is the Reynolds number (a measure for the viscosity of the fluid), $\alpha \in \mathbb{R}\setminus\{0\}$ is a wave number, and the functions $U$, $V$ are the unperturbed velocity profiles. The spectral parameter $\lambda = \mathrm{i}\alpha Rc$

is related to the exponential time dependence of the perturbing stream function $\psi$ and velocity $u$ by

$$\psi(x,y,t) = y_1(x)\exp\bigl(\mathrm{i}\alpha(y-ct)\bigr), \quad u(x,y,t) = y_2(x)\exp\bigl(\mathrm{i}\alpha(y-ct)\bigr);$$

thus $\operatorname{Re}\lambda \geq 0$ corresponds to stability of the perturbation and $\operatorname{Re}\lambda < 0$ to instability. We suppose that $U$ is differentiable, $V$ is twice differentiable, and $U'$, $V$, $V'' \in L_1[0,\infty)\cap L_\infty[0,\infty)$; conditions that are less restrictive but more complicated to describe may be found in [EE87, p. 443].

In the Hilbert space $\mathcal{H}_1 = \mathcal{H}_2 = L_2[0,\infty)$, we define the operators

$$\begin{aligned}
&A := (-\mathrm{D}^2+\alpha^2)^2+\mathrm{i}\alpha RV(-\mathrm{D}^2+\alpha^2)+\mathrm{i}\alpha RV'',\\
&\qquad\qquad \mathcal{D}(A) := \bigl\{y_1 \in W_2^4[0,\infty) : y_1(0) = y_1'(0) = 0\bigr\},\\
&B := 2\mathrm{D}, \qquad \mathcal{D}(B) := W_2^1[0,\infty),\\
&C := 2\mathrm{D} + \mathrm{i}\alpha RU', \qquad \mathcal{D}(C) := W_2^1[0,\infty),\\
&D := -\mathrm{D}^2+\alpha^2 + \mathrm{i}\alpha RV, \qquad \mathcal{D}(D) := \bigl\{y_2 \in W_2^2[0,\infty) : y_2(0) = 0\bigr\},\\
&P := -\mathrm{D}^2+\alpha^2, \qquad \mathcal{D}(P) := \bigl\{y_1 \in W_2^2[0,\infty) : y_1(0) = y_1'(0) = 0\bigr\}.
\end{aligned}$$

The domains for $A$, $C$, and $D$ are appropriately chosen in view of the results in [EE87, p. 443]; in fact, the terms involving $U'$, $V$, $V''$ are relatively compact perturbations of the respective leading terms whenever $U'$, $V$, $V'' \in L_1[0,\infty)\cap L_\infty[0,\infty)$. We introduce the block operator matrices

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) := \mathcal{D}(A)\oplus\mathcal{D}(D),$$

$$\mathcal{B} := \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{D}(\mathcal{B}) := \mathcal{D}(P)\oplus L_2[0,\infty) \subset \mathcal{D}(\mathcal{A}),$$

in the product Hilbert space $\mathcal{H} = L_2[0,\infty)\oplus L_2[0,\infty) = L_2[0,\infty)^2$. If we define the linear operator pencil $\mathcal{L}$ in $L_2[0,\infty)^2$ as

$$\mathcal{L}(\lambda) := \mathcal{A} - \lambda\mathcal{B}, \quad \mathcal{D}\bigl(\mathcal{L}(\lambda)\bigr) := \mathcal{D}(\mathcal{A}), \qquad \lambda \in \mathbb{C},$$

then the Ekman problem is equivalent to the spectral problem $\mathcal{L}(\lambda)y = 0$, $y \in \mathcal{D}(\mathcal{L}(\lambda))$, for the linear pencil $\mathcal{L}$. The essential spectrum of $\mathcal{L}$ is given by

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \bigl\{\lambda \in \mathbb{C} : 0 \in \sigma_{\mathrm{ess}}\bigl(\mathcal{L}(\lambda)\bigr)\bigr\}$$

(see (2.4.6)); so we can use our results to decide whether or not 0 belongs to the essential spectrum of the block operator matrix $\mathcal{L}(\lambda)$ in order to determine the essential spectrum of the linear pencil $\mathcal{L}$.

To this end, we combine two methods presented in Section 2.4: First we split off all relatively compact terms in $\mathcal{L}(\lambda)$ and write $\mathcal{L}(\lambda)$ as a relatively compact perturbation of a block operator matrix $\mathcal{L}_0(\lambda)$ whose entries are differential operators with constant coefficients (see Theorem 2.4.1). In the next step we describe the essential spectrum of $\mathcal{L}_0(\lambda)$ by means of the first Schur complement (see Theorem 2.4.7).

**Proposition 3.2.2** *The block operator matrix $\mathcal{L}(\lambda)$ is diagonally dominant of order* 0 *and closed for all $\lambda \in \mathbb{C}$. Its essential spectrum satisfies*

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}_0) \tag{3.2.3}$$

*where the linear operator pencil $\mathcal{L}_0(\lambda)$, $\lambda \in \mathbb{C}$, in $L_2[0,\infty)^2$ is defined as*

$$\mathcal{L}_0(\lambda) := \begin{pmatrix} (-\mathrm{D}^2+\alpha^2)^2 - \lambda(-\mathrm{D}^2+\alpha^2) & 2\mathrm{D} \\ 2\mathrm{D} & -\mathrm{D}^2+\alpha^2-\lambda \end{pmatrix} =: \begin{pmatrix} A_0-\lambda P & B_0 \\ C_0 & D_0-\lambda \end{pmatrix},$$

$$\mathcal{D}\big(\mathcal{L}_0(\lambda)\big) := \mathcal{D}(A_0) \oplus \mathcal{D}(D_0) := \mathcal{D}(A) \oplus \mathcal{D}(D) = \mathcal{D}\big(\mathcal{L}(\lambda)\big).$$

***Proof.*** Let $\lambda \in \mathbb{C}$ be fixed. The order of the differential operator 2D is strictly less than the orders of the differential operators on the diagonal and its coefficient is bounded on $[0,\infty)$. Thus $C_0$ is $(A_0-\lambda P)$-bounded with relative bound $\delta_{C_0}=0$ and $B_0$ is $(D_0-\lambda)$-bounded with relative bound $\delta_{B_0}=0$ by [EE87, Theorem IX.8.1]. This shows that $\mathcal{L}_0(\lambda)$ is diagonally dominant of order 0 (see Definition 2.2.3). Moreover, it is not difficult to check that $A_0-\lambda P$ and $D_0-\lambda$ are closed (since so are $A_0$ and $D_0$). Hence $\mathcal{L}_0(\lambda)$ is closed by Theorem 2.2.7 i).

If we show that the assumptions i) and ii) of Theorem 2.4.1 are satisfied with $\mathcal{A}_0 = \mathcal{L}_0(\lambda)$ and $\mathcal{A}_1 := \mathcal{L}_1(\lambda) := \mathcal{L}(\lambda) - \mathcal{L}_0(\lambda)$, then $\mathcal{L}_1(\lambda)$ is $\mathcal{L}_0(\lambda)$-compact and all claims follow from Theorem 2.4.1.

For assumption i) we observe that $\delta_{B_0}\delta_{C_0} = 0 < 1$. Since $U'$, $V$, $V'' \in L_1[0,\infty) \cap L_\infty[0,\infty)$, assumption [EE87, (8.13), p. 443] holds and hence, by [EE87, Theorem IX.8.2], the operators $\mathrm{i}\alpha RV(-\mathrm{D}^2+\alpha^2)$, $\mathrm{i}\alpha RV''\cdot$, and $\mathrm{i}\alpha RU'\cdot$ are $(A_0-\lambda P)$-compact and the operator $\mathrm{i}\alpha RV\cdot$ is $(D_0-\lambda)$-compact. Thus also assumption ii) of Theorem 2.4.1 is satisfied. □

**Proposition 3.2.3** *For $\lambda \notin [\alpha^2,\infty)$, we define*

$$\begin{aligned} S_{1,0}(\lambda) &:= A_0 - \lambda P - B_0(D_0-\lambda)^{-1}C_0 \\ &= (-\mathrm{D}^2+\alpha^2)^2 - \lambda(-\mathrm{D}^2+\alpha^2) - 2\mathrm{D}(-\mathrm{D}^2+\alpha^2-\lambda)^{-1}2\mathrm{D}, \\ \mathcal{D}\big(S_{1,0}(\lambda)\big) &:= \mathcal{D}(A_0) = \big\{y_1 \in W_2^4[0,\infty) : y_1(0) = y_1'(0) = 0\big\}. \end{aligned}$$

*Then $S_{1,0}(\lambda)$ is a densely defined closed linear operator and*

$$\sigma_{\mathrm{ess}}(\mathcal{L}_0) \setminus [\alpha^2,\infty) = \sigma_{\mathrm{ess}}(S_{1,0}). \tag{3.2.4}$$

***Proof.*** The operator $D_0 = -\mathrm{D}^2+\alpha^2-\lambda$ with $\mathcal{D}(D_0) = \{y_2 \in W_2^2[0,\infty) : y_2(0) = 0\}$ is boundedly invertible if and only if $\lambda \notin [\alpha^2,\infty)$, and $\sigma(D_0) = \sigma_{\mathrm{ess}}(D_0) = [\alpha^2,\infty)$. Let $\lambda \notin [\alpha^2,\infty)$ be fixed. The value of the first Schur complement of the block operator matrix $\mathcal{L}_0(\lambda)$ at 0 is the operator $S_{1,0}(\lambda)$. Hence, since $\mathcal{L}_0(\lambda)$ is closed, so is $S_{1,0}(\lambda)$ by Theorem 2.2.18. The identity (3.2.4) now follows from Theorem 2.4.7 if we observe that $\lambda \notin [\alpha^2, \infty)$ if and only if $0 \notin [\alpha^2 - \lambda, \infty)$, and hence

$$\begin{aligned} \lambda \in \sigma_{\mathrm{ess}}(\mathcal{L}_0) &\iff 0 \in \sigma_{\mathrm{ess}}\big(\mathcal{L}_0(\lambda)\big) \\ &\iff 0 \in \sigma_{\mathrm{ess}}\big(S_{1,0}(\lambda)\big) \\ &\iff \lambda \in \sigma_{\mathrm{ess}}(S_{1,0}). \end{aligned}$$ □

By means of Propositions 3.2.2 and 3.2.3, the calculation of the essential spectrum of $\mathcal{L}$ has been reduced to the calculation of the essential spectrum of $S_{1,0}$. The latter is performed in the proof of the next theorem.

**Theorem 3.2.4** *The essential spectrum of the linear operator pencil $\mathcal{L}$ associated with the Ekman boundary layer problem is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \big\{\lambda \in \mathbb{C} : \exists\, \xi \in \mathbb{R}\ (\xi^2 + \alpha^2)(\xi^2 + \alpha^2 - \lambda)^2 + 4\xi^2 = 0\big\}.$$

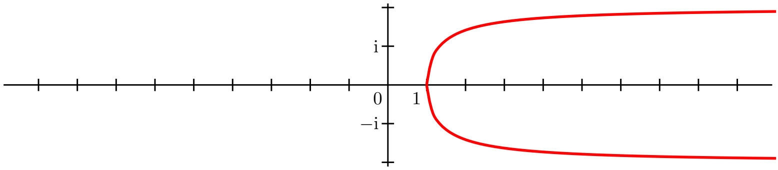



Figure 3.1 Essential spectrum of the Ekman boundary layer problem.

***Proof.*** In order to calculate $\sigma_{\mathrm{ess}}(S_{1,0})$, we define

$$D_1(\lambda) := S_{1,0}(\lambda)(D_0 - \lambda), \quad \lambda \notin [\alpha^2, \infty). \tag{3.2.5}$$

Then $D_1$ is a quadratic operator polynomial, the values $D_1(\lambda)$ are differential operators of order 6 with constant coefficients given by

$$D_1(\lambda) = (-\mathrm{D}^2 + \alpha^2)(-\mathrm{D}^2 + \alpha^2 - \lambda)^2 - 4\mathrm{D}^2,$$
$$\mathcal{D}\big(D_1(\lambda)\big) = \big\{y_1 \in W_2^6[0,\infty) : y_1(0) = y_1''(0) = y_1'''(0) - \alpha^2 y_1'(0) = 0\big\}.$$

We prove that

$$\sigma_{\mathrm{ess}}(S_{1,0}) = \sigma_{\mathrm{ess}}(S_{1,0}) \setminus [\alpha^2, \infty) = \sigma_{\mathrm{ess}}(D_1) \setminus [\alpha^2, \infty). \tag{3.2.6}$$

To this end, let $\lambda \notin [\alpha^2, \infty)$ be fixed. By [GGK90, Theorem XVII.3.1], the product $TS$ of a densely defined Fredholm operator $T$ and a Fredholm operator $S$ is Fredholm. Since $D_0 - \lambda$ and its inverse $(D_0 - \lambda)^{-1}$ are both bijective and hence Fredholm (with index 0) and $S_{1,0}(\lambda)$ is densely defined, [GGK90, Theorem XVII.3.1] together with the relation (3.2.5) implies that $\lambda \notin \sigma_{\mathrm{ess}}(S_{1,0})$ if and only if $\lambda \notin \sigma_{\mathrm{ess}}(D_1)$.

To calculate $\sigma_{\rm ess}(D_1)$, we introduce the (principal) symbol $p_\lambda(\xi)$ of the differential operator $D_1(\lambda)$ for $\lambda \in \mathbb{C}$,

$$p_\lambda(\xi) := (\xi^2 + \alpha^2)(\xi^2 + \alpha^2 - \lambda)^2 + 4\xi^2, \quad \xi \in \mathbb{R}.$$

By well-known results on the essential spectra of differential operators (see [EE87, Corollary IX.9.4]), we have

$$\sigma_{\rm ess}\big(D_1(\lambda)\big) = \big\{p_\lambda(\xi) : \xi \in \mathbb{R}\big\}. \tag{3.2.7}$$

Altogether, (3.2.4), (3.2.6), and (3.2.7) yield

$$\begin{aligned}
&\sigma_{\rm ess}(\mathcal{L}_0) \setminus [\alpha^2, \infty)\\
&= \big\{\lambda \in \mathbb{C} : 0 \in \sigma_{\rm ess}(D_1(\lambda))\big\} \setminus [\alpha^2, \infty)\\
&= \big\{\lambda \in \mathbb{C} : \exists\, \xi \in \mathbb{R}\ (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda)^2 + 4\xi^2 = 0\big\} \setminus [\alpha^2, \infty)\\
&= \big\{\lambda \in \mathbb{C} : \exists\, \xi \in \mathbb{R}\ (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda)^2 + 4\xi^2 = 0\big\} \setminus \{\alpha^2\}.
\end{aligned}$$

It is not difficult to show by explicit calculation that, for every $\lambda \in (\alpha^2, \infty)$, the system of differential equations $\mathcal{L}_0(\lambda)\mathbf{y} = \mathbf{f}$ possesses a unique solution $\mathbf{y} \in \mathcal{D}\big(\mathcal{L}_0(\lambda)\big)$ for every $\mathbf{f} \in L_2[0,\infty)^2$. Hence $\sigma_{\rm ess}(\mathcal{L}_0) \cap (\alpha^2, \infty) = \emptyset$. Since the essential spectrum is closed, and since the formula above shows that there are points of $\sigma_{\rm ess}(\mathcal{L}_0) \setminus [\alpha^2, \infty)$ arbitrarily close to $\alpha^2$, it follows that $\alpha^2 \in \sigma_{\rm ess}(\mathcal{L}_0)$ and so, together with (3.2.3),

$$\sigma_{\rm ess}(\mathcal{L}) = \sigma_{\rm ess}(\mathcal{L}_0) = \big\{\lambda \in \mathbb{C} : \exists\, \xi \in \mathbb{R}\ (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda)^2 + 4\xi^2 = 0\big\}. \quad \square$$

**Remark 3.2.5** The first rigorous proof that the essential spectrum of the Ekman problem has the above form was given by L. Greenberg and M. Marletta in [GM04]; they transformed the problem into a $6\times 6$ first order system of differential equations and constructed explicit singular sequences.

If a component of the complement $\mathbb{C}\setminus\sigma_{\rm ess}(\mathcal{L})$ of the essential spectrum of $\mathcal{L}$ contains a point of the resolvent set $\rho(\mathcal{L})$, then the spectrum in this component consists only of eigenvalues of finite algebraic multiplicity accumulating at most at the essential spectrum (see [GGK90, Theorem XVII.2.1] and Theorem 2.1.10). The complement $\mathbb{C}\setminus\sigma_{\rm ess}(\mathcal{L})$ has two components. In the component containing the left half plane we can choose a sufficiently small point $\lambda^- \in (-\infty, 0)$ so that the distance to the spectrum $[\alpha^2, \infty)$ of $-\mathrm{D}^2+\alpha^2$ is large enough and $\mathcal{L}(\lambda^-)$ is bijective.

It is still an open problem whether there also exists a point $\lambda^+$ in the component in the left half plane containing the interval $(\alpha^2, \infty)$ so that $\mathcal{L}(\lambda^+)$ is bijective; in this case, since the imaginary part of this component is bounded, it is not possible to find a point with sufficiently large distance to $[\alpha^2, \infty)$ (see Fig. 3.1).

## 3.3 Off-diagonally dominant block operator matrices in quantum mechanics

The most prominent off-diagonally dominant block operator matrix arises from the Dirac equation in quantum mechanics. In this section, we apply the results of Chapter 2 to the Dirac equation in $\mathbb{R}^3$ and in curved space time. Our main results include the block diagonalizability of the Dirac operator with electromagnetic potential in $\mathbb{R}^3$ and eigenvalue estimates for the angular part of the Dirac operator in the Kerr-Newman metric.

For the free Dirac operator in $\mathbb{R}^3$, the famous Foldy-Wouthuysen transformation (see [FW50]) yields an explicit transformation of the Dirac operator into block diagonal form (see [Tha88], [Tha92, Chapter 4.3]). In the presence of an electric field, however, such a transformation is not known in closed form. In this case, Foldy and Wouthuysen obtained successive approximations of the required transformation (see *e.g.* [BD64], [CM95]) which, however, do not converge properly. A well-behaved series with limiting block diagonal operator having the same spectrum as the original Dirac operator was established recently by M. Reiher and A. Wolf (see [RW04a]). The existence of an exact transformation for the Dirac operator with electric field was first proved in [LT01] using the results of Section 2.7.

The Dirac equation in the so-called Kerr-Newman metric describes a system of an electrically charged rotating massive black hole and a spin 1/2 particle. It is a remarkable fact that the corresponding system of partial differential equations can be completely separated into a system of ordinary differential equations (see *e.g.* [Pag76], [Cha98]); the differential equations for the angular coordinate $\theta$ and the radial coordinate $r$ are coupled in a complicated way through a parameter that originates from the separation process and acts as the eigenvalue parameter in the angular equation. It is well-known that this so-called angular part of the Dirac operator in the Kerr-Newman metric has discrete spectrum. However, until recently, only numerical approximations of its eigenvalues were known. Analytic eigenvalue estimates were first proved by M. Winklmeier (see [Win05], [Win08]) using the variational principles presented in Section 2.10.

### 3.3.1 Dirac operators in $\mathbb{R}^3$

In this subsection we apply the results of Section 2.7 to the Dirac operator in $\mathbb{R}^3$ with electromagnetic field. First we consider abstract Dirac operators with supersymmetry, which include the free Dirac operator in $\mathbb{R}^3$. Due to the commutation relations therein, the corresponding transformation into block diagonal form in terms of the angular operator $K$ can be determined explicitly. It coincides with the abstract Foldy-Wouthuysen transformation established in [Tha92]; for the free Dirac operator, it reproduces the original Foldy-Wouthuysen transformation from [FW50].

In the second part of this subsection, we consider Dirac operators with electromagnetic potential in $\mathbb{R}^3$. Although in this case no explicit formula for the angular operator $K$ and hence for the transformation is known, our results show that, under certain assumptions on the electromagnetic potential, an exact transformation of the Dirac operator into block diagonal form exists.

**Example 3.3.1** For a relativistic spin 1/2 particle in an external electromagnetic field, the Dirac operator in $\mathbb{R}^3$ is given by

$$\mathbf{H}_{\boldsymbol{\Phi}} := \begin{pmatrix} (mc^2 + e\Phi)I & c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big) \\ c\,\vec{\sigma}\cdot\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big) & (-mc^2 + e\Phi)I \end{pmatrix} \tag{3.3.1}$$

in the space $\mathcal{H} = L_2(\mathbb{R}^3)^2 \oplus L_2(\mathbb{R}^3)^2$. Here $m$ and $e$ are the mass and charge, respectively, of the particle, $c$ is the velocity of light, $\hbar$ the Planck constant, $\vec{\sigma} = (\sigma_1\,\sigma_2\,\sigma_3)$ is the vector of the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\Phi : \mathbb{R}^3 \to \mathbb{R}$ is a potential, and $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector potential. The corresponding electric field $\vec{E}$ and magnetic field $\vec{B}$ are determined by virtue of $\vec{E} = \nabla\Phi$ and $\vec{B} = \operatorname{rot}\vec{A}$ (see *e.g.* [Tha92], [Win00]).

**Definition 3.3.2** Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, let $A$, $D$ be bounded self-adjoint operators in $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, and $B$ a densely defined closed linear operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. Then the self-adjoint block operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(B^*) \oplus \mathcal{D}(B),$$

is called *abstract Dirac operator with supersymmetry* in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ if

$$\begin{aligned} D\big(\mathcal{D}(B)\big) &\subset \mathcal{D}(B), & BD\,y &= -AB\,y, & y &\in \mathcal{D}(B), \\ A\big(\mathcal{D}(B^*)\big) &\subset \mathcal{D}(B^*), & B^*A\,x &= -DB^*x, & x &\in \mathcal{D}(B^*). \end{aligned} \tag{3.3.2}$$

Note that the commutation relations (3.3.2) imply that on $\mathcal{D}(B^*)\oplus\mathcal{D}(B)$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = 0.$$

**Remark 3.3.3** Let $\mathcal{A}$ be an abstract Dirac operator with supersymmetry. If $A$ and $-D$ are positive, we may assume without loss of generality that $B$ and $B^*$ are injective.

In fact, by the supersymmetry assumptions (3.3.2), $\ker B$ is $D$-invariant and $\ker B^*$ is $A$-invariant. Hence the subspaces $\ker B^* \oplus \{0\}$ and $\{0\} \oplus \ker B$ are $\mathcal{A}$-invariant, and we can consider the operator $\mathcal{A}$ in the reduced space $\widetilde{\mathcal{H}} = \big(\mathcal{H}_1 \ominus \ker B^*\big) \oplus \big(\mathcal{H}_2 \ominus \ker B\big)$.

Since $A$ is positive and hence $\big(\mathcal{A}\,(x\ 0)^{\mathrm{t}}, (x\ 0)^{\mathrm{t}}\big) = (Ax, x) > 0$ for all $x \in \mathcal{H}_1$, $x \neq 0$, the subspace $\ker B^* \oplus \{0\}$ belongs to the spectral subspace $\mathcal{L}_+$ of $\mathcal{A}$ corresponding to $(0,\infty)$; analogously, $\{0\} \oplus \ker B$ belongs to the spectral subspace $\mathcal{L}_-$ of $\mathcal{A}$ corresponding to $(-\infty, 0)$. If $\widetilde{\mathcal{L}}_+$ and $\widetilde{\mathcal{L}}_-$ are the spectral subspaces of the reduced operator $\widetilde{\mathcal{A}}$ corresponding to $(0,\infty)$ and $(-\infty, 0)$, respectively, then the spectral subspaces $\mathcal{L}_+$, $\mathcal{L}_-$ of $\mathcal{A}$ are given by the orthogonal sums $\mathcal{L}_+ = \widetilde{\mathcal{L}}_+ \oplus \big(\ker B^* \oplus \{0\}\big)$, $\mathcal{L}_- = \widetilde{\mathcal{L}}_- \oplus \big(\{0\} \oplus \ker B\big)$.

**Theorem 3.3.4** *Let $\mathcal{A}$ be an abstract Dirac operator with supersymmetry in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and let $A$ and $-D$ be positive and boundedly invertible. Then the operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ and its graph $\mathcal{G}(K)$ given by*

$$K := B^*\big(A + (A^2 + BB^*)^{1/2}\big)^{-1}, \quad \mathcal{G}(K) := \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1 \right\} \tag{3.3.3}$$

*have the following properties:*

i) *$K$ is a strict contraction in $\mathcal{H}_1$, $K\big(\mathcal{D}(B^*)\big) \subset \mathcal{D}(B)$, and $K$ satisfies the Riccati equation*

$$\big(KBK + KA - DK - B^*\big)x = 0, \quad x \in \mathcal{D}(B^*).$$

ii) *$\mathcal{G}(K)$ is an $\mathcal{A}$-invariant subspace, i.e. $\mathcal{A}\big(\mathcal{G}(K) \cap \mathcal{D}(\mathcal{A})\big) \subset \mathcal{G}(K)$, and*

$$\mathcal{G}(K) \cap \mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{D}(B^*) \right\};$$

*it coincides with the spectral subspace $\mathcal{L}_+$ of $\mathcal{A}$ corresponding to* $(0,\infty)$.

***Proof.*** i) First we show that $\mathcal{D}(K) = \mathcal{H}_1$ and $K\big(\mathcal{D}(B^*)\big) \subset \mathcal{D}(B)$. Since $A$ is positive and boundedly invertible, so is the operator $A + (A^2 + BB^*)^{1/2}$. By means of quadratic forms (see *e.g.* [Kat95, Sections VI.2.6 and VI.2.7]) and using $\mathcal{D}(B^*) \subset \mathcal{D}(A)$, we see that

$$\mathcal{D}\big((A^2 + BB^*)^{1/2}\big) = \mathcal{D}\big((BB^*)^{1/2}\big) = \mathcal{D}(B^*) = \mathcal{D}\big(A + (A^2 + BB^*)^{1/2}\big)$$

The last equality yields $\mathcal{D}(K) = \mathcal{H}_1$ and, for $x \in \mathcal{D}(B^*)$,

$$\big(A + (A^2 + BB^*)^{1/2}\big)^{-1} x \in \mathcal{D}\Big(\big(A + (A^2 + BB^*)^{1/2}\big)^2\Big) = \mathcal{D}(BB^*),$$

which implies that $Kx \in \mathcal{D}(B)$.

In order to prove that $K$ is a strict contraction, we first show that $A$ commutes with $BB^*$. In fact, if $x \in \mathcal{D}(BB^*)$, then $x \in \mathcal{D}(B^*)$ and $B^*x \in \mathcal{D}(B)$. Now the supersymmetry assumptions (3.3.2) show that $Ax \in \mathcal{D}(B^*)$, $B^*Ax = -DB^*x \in \mathcal{D}(B)$, and $BB^*Ax = -BDB^*x = ABB^*x$. Then $A$ and $A^{1/2}$ also commute with $(A^2+BB^*)^{1/2}$. Now let $x \in \mathcal{H}_1$ be arbitrary and set $y := \bigl(A + (A^2+BB^*)^{1/2}\bigr)^{-1}x$. Since $A$ is positive, it follows that

$$\begin{aligned}
\|x\|^2 &= \bigl\|\bigl(A + (A^2+BB^*)^{1/2}\bigr)y\bigr\|^2 \\
&= \|Ay\|^2 + \bigl((A^2+BB^*)y,y\bigr) + 2\bigl((A^2+BB^*)^{1/2}A^{1/2}y, A^{1/2}y\bigr) \\
&> \bigl((A^2+BB^*)y,y\bigr) > \|B^*y\|^2 = \|Kx\|^2.
\end{aligned}$$

Next we prove that $K$ satisfies the Riccati equation in i). For this, let $x \in \mathcal{D}(B^*)$ and set $y := \bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1}x$. Then $B^*y = Kx \in \mathcal{D}(B)$ and hence $y \in \mathcal{D}(BB^*)$. Since $A$ commutes with $(A^2+BB^*)^{1/2}$, we have

$$\bigl((A^2+BB^*)^{1/2} - A\bigr)\bigl((A^2+BB^*)^{1/2} + A\bigr)y = BB^*y \tag{3.3.4}$$

and hence

$$(A^2+BB^*)^{1/2}x = (A+BK)x. \tag{3.3.5}$$

Using the supersymmetry assumptions (3.3.2), the fact that $A$ commutes with $(A^2+BB^*)^{1/2}$, and (3.3.5), we obtain

$$\begin{aligned}
(B^*+DK)x &= \bigl(B^* - B^*A\bigl(A + (A^2+BB^*)^{1/2}\bigr)^{-1}\bigr)x \\
&= B^*\bigl(I - \bigl(A + (A^2+BB^*)^{1/2}\bigr)^{-1}A\bigr)x \\
&= B^*\bigl(A + (A^2+BB^*)^{1/2}\bigr)^{-1}(A^2+BB^*)^{1/2}x \\
&= K\bigl(A+BK\bigr)x.
\end{aligned}$$

ii) The $\mathcal{A}$-invariance of $\mathcal{G}(K)$ is immediate from the Riccati equation in i) (compare the proof of Proposition 2.9.12). The formula for $\mathcal{G}(K) \cap \mathcal{D}(\mathcal{A})$ follows from the inclusion $K\bigl(\mathcal{D}(B^*)\bigr) \subset \mathcal{D}(B)$ proved in i). In order to show that $\mathcal{G}(K) = \mathcal{L}_+ = \mathcal{L}_{(0,\infty)}(\mathcal{A})$, it suffices to prove that

$$(\mathcal{A}\mathbf{x},\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathcal{G}(K) \cap \mathcal{D}(\mathcal{A}), \tag{3.3.6}$$

$$(\mathcal{A}\mathbf{x},\mathbf{x}) \leq 0, \quad \mathbf{x} \in \mathcal{G}(K)^\perp \cap \mathcal{D}(\mathcal{A}), \quad \mathcal{G}(K)^\perp = \left\{\begin{pmatrix} -K^*y \\ y \end{pmatrix} : y \in \mathcal{H}_2\right\}. \tag{3.3.7}$$

Using the Riccati equation, we see that, for $x \in \mathcal{D}(B^*)$,

$$\begin{aligned}
\left(\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}\begin{pmatrix} x \\ Kx \end{pmatrix}, \begin{pmatrix} x \\ Kx \end{pmatrix}\right) &= \bigl((I+K^*K)(A+BK)x, x\bigr) \\
&= \bigl((A + K^*KA + BK + K^*KBK)x, x\bigr).
\end{aligned}$$

The operator $A$ is positive by assumption. Since $A$ commutes with $(A^2+BB^*)^{1/2}$ and by the supersymmetry assumptions (3.3.2), we have

$$\begin{aligned} KA &= B^*\bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1}A = B^*A\bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1} \\ &= -DB^*\bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1} = -DK. \end{aligned}$$

Hence the operator $K^*KA = -K^*DK$ is non-negative on $\mathcal{D}(B^*)$ since $-D$ is positive. The operator $BK$ is non-negative on $\mathcal{D}(B^*)$ since, by (3.3.5),

$$BK = \bigl((A^2+BB^*)^{1/2}-A\bigr) \geq 0.$$

Finally, the operator $K^*KBK$ is non-negative because of the inequality

$$KB = B^*\bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1}B \geq 0.$$

This completes the proof of (3.3.6). For the proof of (3.3.7) we observe that, by the supersymmetry conditions (3.3.2), for $x\in\mathcal{D}(B)$ we have $D^2x\in\mathcal{D}(B)$ and $B(D^2+B^*B)x = (A^2+BB^*)Bx$. By the assumptions on $A$ and $-D$, the operators $D^2+B^*B$, $A^2+BB^*$ are boundedly invertible and so

$$(A^2+BB^*)^{-n}Bx = B(D^2+B^*B)^{-n}x, \quad n\in\mathbb{N}.$$

Since the square root can be approximated by polynomials, this implies that $(A^2+BB^*)^{-1/2}B = B(D^2+B^*B)^{-1/2}$ and hence, again by (3.3.2),

$$\bigl(A+(A^2+BB^*)^{1/2}\bigr)Bx = B\bigl(-D+(D^2+B^*B)^{1/2}\bigr)x.$$

Altogether, we have shown that, for $x\in\mathcal{D}(B)$,

$$K^*x = \bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1}Bx = B\bigl(-D+(D^2+B^*B)^{1/2}\bigr)^{-1}x.$$

Since $K$ and hence $K^*$ are bounded and $\mathcal{D}(B)$ is dense in $\mathcal{H}_2$, it follows that

$$K^* = B\bigl(-D+(D^2+B^*B)^{1/2}\bigr)^{-1}.$$

This formula is completely analogous to the formula for $K$, with $A$ replaced by $-D$ and $B$ by $B^*$. Hence (3.3.7) follows in a similar way as (3.3.6). □

Since the abstract Dirac operator $\mathcal{A}$ from Theorem 3.3.4 satisfies the assumptions of Theorem 2.7.7, the operator $K$ given by (3.3.3) is, in fact, the angular operator established in Theorem 2.7.7. Using Corollary 2.7.8, we now obtain an explicit formula for the unitary operator transforming $\mathcal{A}$ into block diagonal form.

**Corollary 3.3.5** *Let $\mathcal{A}$ be an abstract Dirac operator with supersymmetry in $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ with $A$ and $-D$ positive and boundedly invertible. Set*

$$K := B^*\bigl(A+(A^2+BB^*)^{1/2}\bigr)^{-1}, \quad K^* := B\bigl(-D+(D^2+B^*B)^{1/2}\bigr)^{-1}$$

*and define*

$$\mathcal{U} := \begin{pmatrix} I & -K^* \\ K & I \end{pmatrix} \begin{pmatrix} (I+K^*K)^{-1/2} & 0 \\ 0 & (I+KK^*)^{-1/2} \end{pmatrix}.$$

*Then $\mathcal{U}$ is a unitary operator in $\mathcal{H}$ and on $\mathcal{D}(B^*) \oplus \mathcal{D}(B)$ we have*

$$\mathcal{U}^{-1} \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \mathcal{U} = \begin{pmatrix} (A^2+BB^*)^{1/2} & 0 \\ 0 & -(D^2+B^*B)^{1/2} \end{pmatrix}.$$

***Proof.*** Since $K$ is the angular operator from Theorem 2.7.7, the equality $\mathcal{U}^*\mathcal{U} = I$ follows from (2.8.2); the proof for $\mathcal{U}^*\mathcal{U}$ is similar. If we set $\Gamma_+ := (I+K^*K)^{-1/2}$, $\Gamma_- := (I+KK^*)^{-1/2}$, then Theorem 2.8.1 yields

$$\mathcal{U}^{-1} \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \mathcal{U} = \begin{pmatrix} \Gamma_+ & 0 \\ 0 & \Gamma_- \end{pmatrix}^{-1} \begin{pmatrix} A+BK & 0 \\ 0 & D-K^*B^* \end{pmatrix} \begin{pmatrix} \Gamma_+ & 0 \\ 0 & \Gamma_- \end{pmatrix};$$

note that Corollary 2.7.25 together with the inclusion $K\big(\mathcal{D}(B^*)\big) \subset \mathcal{D}(B)$ implies that $\mathcal{D}_+ = \mathcal{D}(B^*)$, $\mathcal{D}_- = \mathcal{D}(B)$. By (3.3.5), we have

$$A + BK = (A^2+BB^*)^{1/2};$$

analogously, one can show that $D - K^*B^* = -(D^2+B^*B)^{1/2}$. It remains to be proved that *e.g.* $\Gamma_+^{-1}(A+BK)\Gamma_+ = (A^2+BB^*)^{1/2}$, that is, $\Gamma_+$ commutes with $(A^2+BB^*)^{1/2}$. From (3.3.4) it follows that

$$BB^* = \big(A + (A^2+BB^*)^{1/2}\big)^2 - 2A\big(A + (A^2+BB^*)^{1/2}\big)$$

and hence

$$\begin{aligned} I + K^*K &= I + \big(A+(A^2+BB^*)^{1/2}\big)^{-1} BB^* \big(A+(A^2+BB^*)^{1/2}\big)^{-1} \\ &= 2\Big(I - A\big(A+(A^2+BB^*)^{1/2}\big)^{-1}\Big). \end{aligned}$$

Using the Neumann series for $\Gamma_+^2 = (I+K^*K)^{-1}$, one can now deduce that $\Gamma_+$ commutes with $(A^2+BB^*)^{1/2}$. □

**Remark 3.3.6** The Dirac operator $\mathbf{H_0}$ in $\mathbb{R}^3$ without electric potential, *i.e.* $\Phi = 0$ in (3.3.1), satisfies the supersymmetry assumptions (3.3.2). According to the previous corollary, it is unitarily equivalent to the operator

$$\begin{pmatrix} \big(m^2c^4 + c^2\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big)^2 - ec\,\vec{\sigma}\cdot\vec{B}\big)^{1/2} & 0 \\ 0 & -\big(m^2c^4 + c^2\big(-\mathrm{i}\hbar\nabla - \frac{e}{c}\vec{A}\big)^2 - ec\,\vec{\sigma}\cdot\vec{B}\big)^{1/2} \end{pmatrix}.$$

The corresponding unitary transformation in Corollary 3.3.5 is called *Foldy-Wouthuysen transformation* in quantum mechanics (see [Tha92, Theorem 5.13]). In this case, the subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ from Theorem 3.3.4 correspond to the so-called *electronic* and *positronic subspace*, respectively.

If the potential $\Phi$ is not a constant, the Dirac operator $\mathbf{H}_{\boldsymbol{\Phi}}$ is no longer supersymmetric. Nevertheless, the existence of a transformation bringing it into block diagonal form is guaranteed *e.g.* if the norm of the potential is sufficiently small; it should be possible to extend this result also to relatively bounded potentials that do not close the spectral gap.

**Theorem 3.3.7** *Suppose that $\vec{A}$ is $\mathrm{i}\hbar\nabla$-bounded with relative bound $< 1$ and that $\Phi$ is a bounded multiplication operator in $L_2(\mathbb{R}^3)^2$ with*

$$\|e\Phi\| \le mc^2. \tag{3.3.8}$$

*Then the Dirac operator $\mathbf{H}_{\boldsymbol{\Phi}}$ is self-adjoint on $H^1(\mathbb{R}^3)^4 = H^1(\mathbb{R}^3)^2 \oplus H^1(\mathbb{R}^3)^2$ where $H^1(\mathbb{R}^3)$ is the first order Sobolev space. If we set*

$$\begin{aligned} \mathcal{L}_+ &:= \ \mathcal{L}_{(0,\infty)} \dotplus \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in H^1(\mathbb{R}^3)^4 : \exists\, y \in H^1(\mathbb{R}^3)^2 \ \begin{pmatrix} x \\ y \end{pmatrix} \in \ker \mathbf{H}_{\boldsymbol{\Phi}} \right\}, \\ \mathcal{L}_- &:= \mathcal{L}_{(-\infty,0)} \dotplus \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \in H^1(\mathbb{R}^3)^4 : \exists\, x \in H^1(\mathbb{R}^3)^2 \ \begin{pmatrix} x \\ y \end{pmatrix} \in \ker \mathbf{H}_{\boldsymbol{\Phi}} \right\}, \end{aligned}$$

*then $H^1(\mathbb{R}^3)^4 = \mathcal{L}_+ \dotplus \mathcal{L}_-$ and the subspaces $\mathcal{L}_\pm$ have representations*

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in L_2(\mathbb{R}^3)^2 \right\}, \quad \mathcal{L}_- = \left\{ \begin{pmatrix} -K^* y \\ y \end{pmatrix} : y \in L_2(\mathbb{R}^3)^2 \right\}$$

*with a contraction $K : L_2(\mathbb{R}^3)^2 \to L_2(\mathbb{R}^3)^2$. On the subspaces*

$$\begin{aligned} \mathcal{D}_+ &:= \left\{ x \in H^1(\mathbb{R}^3)^2 : \ Kx \in H^1(\mathbb{R}^3)^2 \right\}, \\ \mathcal{D}_- &:= \left\{ y \in H^1(\mathbb{R}^3)^2 : K^* y \in H^1(\mathbb{R}^3)^2 \right\}, \end{aligned}$$

*the Dirac operator $\mathbf{H}_{\boldsymbol{\Phi}}$ admits the block diagonalization*

$$\begin{aligned} &\mathcal{V}^{-1} \begin{pmatrix} mc^2 + e\Phi & c\,\vec{\sigma}\cdot\bigl(-i\hbar\nabla - \frac{e}{c}\vec{A}\bigr) \\ c\,\vec{\sigma}\cdot\bigl(-i\hbar\nabla - \frac{e}{c}\vec{A}\bigr) & -mc^2 + e\Phi \end{pmatrix} \mathcal{V} \\ &= \begin{pmatrix} mc^2 + e\Phi + c\,\vec{\sigma}\cdot\bigl(-i\hbar\nabla - \frac{e}{c}\vec{A}\bigr)K & 0 \\ 0 & -mc^2 + e\Phi + c\,\vec{\sigma}\cdot\bigl(-i\hbar\nabla - \frac{e}{c}\vec{A}\bigr)K^* \end{pmatrix}, \end{aligned}$$

*where the transformation matrix is given by*

$$\mathcal{V} := \begin{pmatrix} I & -K^* \\ K & I \end{pmatrix}.$$

*If the norm inequality (3.3.8) is strengthened to $\|e\Phi\| < mc^2$, then*

$$\bigl(-mc^2 + \|e\Phi\|\,, mc^2 - \|e\Phi\|\bigr) \subset \rho(\mathbf{H}_{\boldsymbol{\Phi}}),$$

*and the contraction $K$ is uniform.*

***Proof.*** The self-adjointness of $\mathbf{H}_\Phi$ follows from the Kato-Rellich Theorem (see [Kat95, Theorem V.4.3]). All other statements follow from Theorem 2.7.7 together with Remark 2.7.12 and from Theorem 2.8.1. □

**Remark 3.3.8** The assumption on $\vec{A}$ is satisfied *e.g.* if, for some $p > 3$, $\vec{A} \in L_p(\mathbb{R}^3) + L_\infty(\mathbb{R}^3)$ (see [Pro63]).

### 3.3.2 The angular part of the Dirac equation in the Kerr-Newman metric

The Dirac equation in the so-called Kerr-Newman metric describes an electrically charged rotating massive black hole. Separation of variables in the corresponding system of partial differential equations leads to two systems of ordinary differential equations for the angular coordinate $\theta$ and the radial coordinate $r$; the coupling parameter $\lambda$ is the spectral parameter for the angular part and appears as a parameter in the radial spectral problem. The latter has essential spectrum covering the whole real axis (see [BM99]), whereas the angular part has discrete spectrum (see [Win05] and below). Any knowledge about the eigenvalues of the angular part may be useful to decide whether or not the full Dirac equation in the Kerr-Newman metric possesses eigenvalues.

In this section, we investigate the spectral problem of the angular part of the Dirac equation in the Kerr-Newman metric. The corresponding system of differential equations for the angular coordinate $\theta \in (0,\pi)$ is of the form

$$\left(\begin{pmatrix} -am\cos\theta & \dfrac{\mathrm{d}}{\mathrm{d}\theta} + \dfrac{k+1/2}{\sin\theta} + a\omega\sin\theta \\ -\dfrac{\mathrm{d}}{\mathrm{d}\theta} + \dfrac{k+1/2}{\sin\theta} + a\omega\sin\theta & am\cos\theta \end{pmatrix} - \lambda\right)\Psi = 0$$

where $a \in \mathbb{R}$ is the angular momentum per unit mass of the black hole, $m \geq 0$ is the mass of the spin 1/2 particle, $k \in \mathbb{Z}$ is related to the motion perpendicular to the symmetry axis, and $\omega \in \mathbb{R}$ is the energy of the particle.

**Lemma 3.3.9** *In the Hilbert space $L_2(0,\pi)$, we define the operators*

$$A := -am\cos(\,\cdot\,)\,\cdot\,, \quad \mathcal{D}(A) := L_2(0,\pi),$$

*and, for $k \in \mathbb{Z}$,*

$$B_k := \frac{\mathrm{d}}{\mathrm{d}\theta} + \frac{k+1/2}{\sin(\,\cdot\,)} + a\omega\sin(\,\cdot\,),$$

$$\mathcal{D}(B_k) := \left\{ g \in L_2(0,\pi) : g \text{ absolutely continuous},\ g' + \frac{k+1/2}{\sin(\,\cdot\,)}\, g \in L_2(0,\pi) \right\}.$$

*Then $A$ is bounded and self-adjoint, $\|A\| = |am|$ and $\sigma(A) = \big[-|am|, |am|\big]$. The operator $B_k$ is boundedly invertible and has compact resolvent. The eigenvalues $\nu_n(B_kB_k^*)$, $n \in \mathbb{N}$, of $B_kB_k^*$ satisfy the two-sided estimates*

$$\begin{aligned} \nu_n(B_kB_k^*) &\geq \nu_n^-(B_kB_k^*) := \max\big\{0, \big(|k+1/2| - 1/2 + n\big)^2 + \Omega_-\big\}, \\ \nu_n(B_kB_k^*) &\leq \nu_n^+(B_kB_k^*) := \big(|k+1/2| - 1/2 + n\big)^2 + \Omega_+ \end{aligned} \tag{3.3.9}$$

*with*

$$\begin{aligned} \Omega_- &:= 2(k+1/2)\, a\omega - |a\omega|, \\ \Omega_+ &:= \begin{cases} a^2\omega^2 + 1/4 + 2(k+1/2)\, a\omega & \text{if } \ 2\, a\omega \notin [-1,\, 1\,], \\ 2(k+1/2)\, a\omega + |a\omega| & \text{if } \ 2\, a\omega \in [-1,\, 1\,]. \end{cases} \end{aligned}$$

***Proof.*** The assertions for $A$ are obvious. That $B_k$ is closed follows from the theory of ordinary differential equations (*e.g.* noting that $B_k$ is the adjoint of the differential operator in the other off-diagonal corner); the estimates (3.3.9) follow with the help of Sturm's comparison theorem (see [Win05] for details). □

**Remark 3.3.10** If $a = 0$, then $\Omega_- = \Omega_+ = 0$. Then the upper and lower bound in the estimates (3.3.9) coincide and yield the true eigenvalues $\nu_n(B_kB_k^*) = \big(|k+1/2| - 1/2 + n\big)^2$.

**Theorem 3.3.11** *The operator of the angular part of the Dirac equation in the Kerr-Newman metric is given by the block operator matrix*

$$\mathbf{H}_{\mathrm{ang}} := \begin{pmatrix} A & B_k \\ B_k^* & -A \end{pmatrix}, \quad \mathcal{D}(\mathbf{H}_{\mathrm{ang}}) := \mathcal{D}(B_k^*) \oplus \mathcal{D}(B_k),$$

*in $L_2(0,\pi) \oplus L_2(0,\pi)$ with the operators $A$, $B_k$ defined as in Lemma* 3.3.9. *It is self-adjoint and has compact resolvent. The eigenvalues of $\mathbf{H}_{\mathrm{ang}}$ form two sequences $(\lambda_n)_{n\in\mathbb{N}}$, $(\lambda_{-n})_{n\in\mathbb{N}}$ with $\lambda_n \to \infty, \lambda_{-n} \to -\infty$ if $n \to \infty$.*

***Proof.*** We decompose

$$\mathbf{H}_{\mathrm{ang}} = \begin{pmatrix} 0 & B_k \\ B_k^* & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} =: \mathcal{S} + \mathcal{T}. \tag{3.3.10}$$

Then $\mathcal{S}$ is self-adjoint with compact resolvent since $B_k$ is closed (see Proposition 2.6.3) with compact resolvent, and $\mathcal{T}$ is bounded and self-adjoint; the eigenvalues of $\mathcal{S}$ are the two sequences $\big(\nu_n(B_kB_k^*)\big)_{n\in\mathbb{N}}$, $\big(-\nu_n(B_kB_k^*)\big)_{n\in\mathbb{N}}$. Hence standard perturbation arguments (see [Kat95, Theorems IV.3.17, V.4.3, V.4.10]), together with the estimates (3.3.9), yield all claims. □

In the following we give two different eigenvalue estimates for $\mathbf{H}_{\rm ang}$. First we apply the variational principle from Theorem 2.10.4 to obtain upper and lower bounds for all eigenvalues of $\mathbf{H}_{\rm ang}$ (see [Win05, Theorem 4.32]), [Win08, Theorem 5.4]). This estimate only depends on the quantity $am$; it may also be obtained by perturbation arguments for the decomposition (3.3.10). For the smallest eigenvalue in modulus, we derive another estimate in terms of the product $a\omega$ by means of a unitary transformation of $\mathbf{H}_{\rm ang}$ and arguments similar to those of Theorem 2.5.18 and Remark 2.5.19 on the spectral gap (see [Win05, Theorem 5.10]).

**Theorem 3.3.12** *Let $(\lambda_n)_{n=m_++1}^{\infty}$ with $m_+ \in \mathbb{N}_0$ be the increasing sequence of eigenvalues of $\mathbf{H}_{\rm ang}$ in the interval $\big(|am|,\infty\big)$. Moreover, let the functionals $\lambda_+(x,y)$ be defined as in formula* (2.10.5) *for $x \in \mathcal{D}(B_k^*)$, $y \in \mathcal{D}(B_k)$, and let $S_1(\lambda) = A - \lambda + B_k(A+\lambda)B_k^*$, $\lambda > |am|$, be the first Schur complement of $\mathbf{H}_{\rm ang}$ as defined in Proposition* 2.10.1. *Then*

$$\kappa := \kappa_-(\gamma) = \dim \mathcal{L}_{(-\infty,0)} S_1(\gamma) < \infty$$

*for all $\gamma \in \big(|am|, \lambda_{m_++1}\big)$ and the eigenvalues of $\mathbf{H}_{\rm ang}$ satisfy*

$$\lambda_{m_++n} = \min_{\substack{\mathcal{L}\subset\mathcal{D}(B_kB_k^*)\\ \dim\mathcal{L}=m_++\kappa}} \max_{\substack{x\in\mathcal{L}\\ x\neq 0}} \max_{\substack{y\in\mathcal{D}(B_k)\\ y\neq 0}} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}; \tag{3.3.11}$$

*in terms of the physical parameters $a, m, \omega$, they can be estimated by*

$$\sqrt{\nu_{\kappa+n}^-(B_kB_k^*)} - |am| \le \lambda_{m_++n} \le \sqrt{\nu_{\kappa+n}^+(B_kB_k^*)} + |am|; \tag{3.3.12}$$

*here $\nu_{n+n_0}^{\pm}(B_kB_k^*)$ are the bounds for the eigenvalues of $B_kB_k^*$ in* (3.3.9).

***Proof.*** For $\lambda > |am|$, the operator $B_k(A+\lambda)B_k^*$ is positive and hence $\dim \mathcal{L}_{(-\infty,0)}\big(B_k(A+\lambda)B_k^*\big) = 0$. Moreover, since $B_k(A+\lambda)B_k^*$ has compact resolvent by Lemma 3.3.9 and $A-\lambda$ is bounded, it follows that $\dim \mathcal{L}_{(-\infty,0)}\big(A-\lambda+B_k(A+\lambda)B_k^*\big) < \infty$. Thus the index shift $\kappa$ is finite. Now formula (3.3.11) follows from Theorem 2.10.4; the estimate (3.3.12) is a consequence of Theorem 2.11.5 and of the estimates (3.3.9). □

If the quantity $|am|$ is large, the above eigenvalue bounds are of limited use. In this case, the following alternative estimate may give a better bound for the smallest eigenvalue in modulus.

**Theorem 3.3.13** *Assume that the physical constants $a, \omega$ satisfy*

$$\operatorname{sign}(k+1/2)\, a\omega \ge -|k+1/2|.$$

*Then every eigenvalue $\lambda$ of $\mathbf{H}_{\rm ang}$ satisfies $|\lambda| > \lambda_{\rm gap}$ with*

$$\lambda_{\rm gap} := \begin{cases} \operatorname{sign}(k+1/2)\big(a\omega+k+1/2\big) & \textit{if} \;\; |a\omega| \geq |k{+}1/2|, \\ 2\sqrt{a\omega(k+1/2)} & \textit{if} \;\; \operatorname{sign}(k{+}1/2)\, a\omega > |k{+}1/2|. \end{cases}$$

***Proof.*** The block operator matrix

$$\mathcal{U} := \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix}$$

defines a unitary transformation of $L_2(0,\pi) \oplus L_2(0,\pi)$. If we define the formal differential expressions

$$\tau_A := \frac{k{+}1/2}{\sin(\,\cdot\,)} + a\omega\sin(\,\cdot\,), \quad \tau_B := \frac{\mathrm{d}}{\mathrm{d}\theta} + am\cos(\,\cdot\,), \quad \tau_B^* := -\frac{\mathrm{d}}{\mathrm{d}\theta} + am\cos(\,\cdot\,),$$

then the operator $\mathbf{H}_{\rm ang}^{\mathcal{U}} := \mathcal{U}\,\mathbf{H}_{\rm ang}\,\mathcal{U}^{-1}$ with domain $\mathcal{D}(\mathbf{H}_{\rm ang}^{\mathcal{U}}) = \mathcal{U}\,\mathcal{D}(\mathbf{H}_{\rm ang})$ is given by

$$\begin{aligned} D\big(\mathbf{H}_{\rm ang}^{\mathcal{U}}\big) &= \left\{ \begin{pmatrix} f+g \\ f-g \end{pmatrix} : f \in \mathcal{D}(B^*),\, g \in \mathcal{D}(B) \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in L_2(0,\pi)^2 : \tau_A x + \tau_B y \in L_2(0,\pi), \tau_B^* x - \tau_A y \in L_2(0,\pi) \right\}, \end{aligned}$$

$$\mathbf{H}_{\rm ang}^{\mathcal{U}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tau_A x + \tau_B y \\ \tau_B^* x - \tau_A y \end{pmatrix};$$

note that, in general, $\mathbf{H}_{\rm ang}^{\mathcal{U}}$ is no longer a block operator matrix since its domain may not decouple.

Let $\lambda \in \sigma_{\rm p}(\mathbf{H}_{\rm ang}) = \sigma_{\rm p}(\mathbf{H}_{\rm ang}^{\mathcal{U}})$ be an eigenvalue with corresponding eigenfunction $(x\ y)^{\rm t} \in \mathcal{D}(\mathcal{A}_U)$. We can show that the scalar products $(\tau_A x, x)$, $(\tau_A y, y)$, and $(\tau_B y, x) = (\tau_B^* x, y)$ exist and hence

$$(\tau_A x, x) - \lambda\|x\|^2 = -(\tau_B y, x), \quad -(\tau_A y, y) - \lambda\|y\|^2 = -(\tau_B^* x, y). \quad (3.3.13)$$

Now assume $k+1/2 > 0$; the case $k+1/2 < 0$ is analogous. It is not hard to see that

$$\frac{k+1/2}{\sin\theta} + a\omega\sin\theta \geq \lambda_{\rm gap} \geq 0, \quad \theta \in (0,\pi),$$

if $\operatorname{sign}(k+1/2)\, a\omega \geq -|k+1/2|$, and $\lambda_{\rm gap}$ is attained in at most one point $\theta \in (0,\pi)$. Thus $(\tau_A x, x) > \lambda_{\rm gap}\|x\|^2$ for all $x \in L_2(0,\pi)$ for which the scalar product exists. In both equations in (3.3.13), the left hand side is real and hence so are the right hand sides. This implies $(\tau_B y, x) = \overline{(\tau_B y, x)} = (x, \tau_B y) = (\tau_B^* x, y)$. Hence, subtracting the equations in (3.3.13), we obtain

$$\lambda\left(\|x\|^2 - \|y\|^2\right) = (\tau_A x, x) + (\tau_A y, y) > \lambda_{\rm gap}\left(\|x\|^2 + \|y\|^2\right) \geq 0. \quad (3.3.14)$$

It follows that $\lambda \neq 0$ and $\|x\| \neq \|y\|$, and that $\lambda > 0$ if and only if $\|x\| > \|y\|$.

If $\lambda > 0$, then

$$\lambda > \lambda_{\text{gap}} \frac{\|x\|^2 + \|y\|^2}{\|x\|^2 - \|y\|^2} \geq \lambda_{\text{gap}}.$$

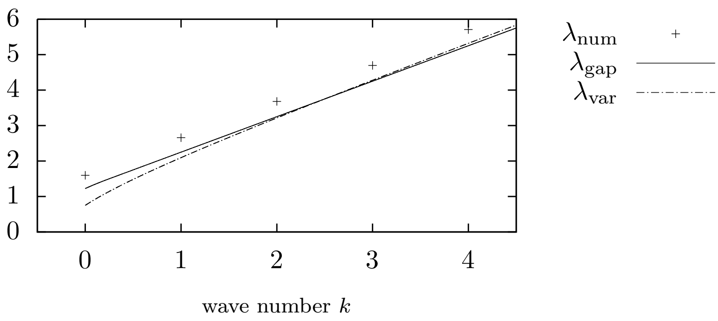
If $\lambda < 0$, then $\|x\|^2 - \|y\|^2 < 0$ and

$$\lambda < \lambda_{\text{gap}} \frac{\|x\|^2 + \|y\|^2}{\|x\|^2 - \|y\|^2} \leq -\lambda_{\text{gap}}. \qquad \square$$

**Remark 3.3.14** Numerical approximations for the eigenvalues of $\mathbf{H}_{\text{ang}}$ were calculated by K.G. Suffern, E.D. Fackerell, and C.M. Cosgrove, and by S.K. Chakrabarti (see [SFC83], [Cha84]). In [SFC83] the authors employed a series ansatz for the eigenfunctions in terms of hypergeometric functions and obtained a three term recurrence relation for the coefficients. They expanded the eigenvalues in terms of $a(m - \omega)$ and $a(m + \omega)$ and approximated them numerically for $-5 \leq k \leq 4$ and two fixed pairs of values of $am$ and $a\omega$. In Table 3.1 the numerical approximations $\lambda_{\text{num}}$ from [SFC83] for the first eigenvalue are compared to the corresponding lower bounds $\lambda_{\text{var}}$ from Theorem 3.3.12 and $\lambda_{\text{gap}}$ from Theorem 3.3.13 for $0 \leq k \leq 4$, $am = 0.25$, and $a\omega = 0.75$; note that Theorem 3.3.13 applies since $a\omega = 0.25 \geq |k + 1/2|$. Moreover, one can show that $\kappa = 0$ and $m_+ = 0$ in Theorem 3.3.12.

Table 3.1 Numerical values and analytic lower bounds for the first positive eigenvalue in the case $am = 0.25$ and $a\omega = 0.75$.

| k | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\lambda_{\text{num}}$ | 1.59764 | 2.65654 | 3.68229 | 4.69685 | 5.70622 |
| $\lambda_{\text{var}}$ | 0.75000 | 2.09521 | 3.21410 | 4.27769 | 5.31776 |
| $\lambda_{\text{gap}}$ | 1.22474 | 2.25000 | 3.25000 | 4.25000 | 5.25000 |



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